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THE NORWEGIAN NORTH POLAR EXPEDITION WITH THE "MAUD"
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CONTRIBUTION TO THE DYNAMICS OF FREE PROGRESSIVE TIDAL WAVES

BY

JONAS EKMAN FJELDSTAD

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Introduction.

The following theory can primarily be applied to the tides of the Arctic Ocean and especially on the North Siberian shelf.

The "Maud-Expedition" had during the years 1918—1921 made tidal observations at Cape Chelyuskin, Ayon-Island and Cape Serdze Kamen. These observations were sent to the Geophysical Institute, Bergen, where I was intrusted with their computation and wrote a preliminary report on the results in "Naturen" (June—August 1923).

It was found that the semi-diurnal tides in the Arctic must be a free wave propagating across the Arctic Ocean from the North Atlantic, and that the tides on the North Siberian shelf must be strongly affected by frictional resistance, and I supposed that the ice-cover on the surface must be one of the principal reasons for this great frictional resistance.

As I later started the final discussion of the observations, I realized that a study of the influence of the eddy viscossity on free tidal waves would be of interest when trying to explain the tidal phenomena on the North Siberian shelf.

As I had performed the integration of the equations of motion in the case of constant eddy viscossity, the treatise of H. U. Sverdrup "Dynamic of Tides on the North Siberian shelf",[1]) came into my hands, and I then saw that he had dealt with the same problem and that he had utilized his theory on some of his rich material of current-measurements. He was able to explain the most peculiar features of the currents as caused by the combined effect of frictional resistance and the deflecting force of the earth's rotation. However, his investigations were in some respects incomplete, because he had only dealt with the case where the period of the wave is shorter than one pendulum-day. As the results which I had found in some ways went further, I decided to continue the investigations which supplement and carry on the results found by Sverdrup. Furthermore I have investigated the influence of an eddy viscossity which varies with the depth, and lastly I have calculated a few numerical examples in order to compare the theory with observations.

Professor Sverdrup has kindly placed at my disposal the observation-material and has also taken a great interest in my work, for which I take the opportunity of giving him my best thanks. I have, however, not utilized much of the material in the

[1]) Geofysiske publikationer, vol. IV, no. 5.

present paper, as I intend to use the theoretical results here found in the final discussion of the complete tidal observations from the "Maud".

I also take the opportunity to express my sincerest thanks to the Director of the Geophysical Institute Professor Helland-Hansen, who has made it possible for me to carry on these investigations, and has been constantly encouraging me during the work.

1. General remarks.

The tidal waves which we are going to treat in this paper are mainly the waves whose periods are of about 12 hours. The validity of the theory is not restricted to these waves, but in most cases the results shall be applied to the semi-diurnal waves. The most important of the semi-diurnal waves is the lunar component M_2 whose period is $12^h\ 25^m$, and it is this wave which generally gives the tidal phenomena their character.

In the oceans the tidal waves appear as a result of forced oscillations under the influence of the tide-producing potential, and waves propagated from adjacent waters and having the character of standing waves or free progressive waves.

In the Polar Ocean the forced oscillations are relatively small, as the tide-producing potential vanishes at the pole. Practically the whole semi-diurnal wave is propagated from the North Atlantic. In the deep polar-basin the semi-diurnal wave may probably be considered as constituted of a standing wave and a progressive wave; but on the broad continental shelf north of Siberia the damping is so considerable that almost only the progressive wave remains. The study of progressive waves forms the subject of this paper.

The hydrodynamical equations of long waves on a rotating globe are[1])

$$\frac{\partial u}{\partial t} - 2\omega v \cos\theta = -\frac{g}{a}\frac{\partial \zeta}{\partial \theta}$$

$$\frac{\partial v}{\partial t} + 2\omega u \cos\theta = -\frac{g}{a \sin\theta}\frac{\partial \zeta}{\partial \psi}$$

$$\frac{\partial \zeta}{\partial t} = -\frac{1}{a \sin\theta}\left(\frac{\partial}{\partial \theta}(h u \sin\theta) + \frac{\partial}{\partial \psi}(h v)\right),$$

where

$$u = a\frac{d\theta}{dt}, \quad v = a \sin\theta \frac{d\psi}{dt},$$

θ is the colatitude and ψ the longitude, a the mean radius of the earth and ω the angular velocity of the earth $= 0.7292 \cdot 10^{-4}$. ζ is the elevation of the surface and h the depth.

If u_1 and v_1 denote the velocities along and perpendicular to any horizontal direction x, we easily find by transformation of the co-ordinates

$$\frac{\partial u_1}{\partial t} - 2\omega v_1 \cos\theta = -g\frac{\partial \zeta}{\partial x}$$

$$\frac{\partial v_1}{\partial t} + 2\omega u_1 \cos\theta = -g\frac{\partial \zeta}{\partial y}$$

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}(h u_1) - \frac{\partial}{\partial y}(h v_1).$$

[1]) Lamb: Hydrodynamics, p. 320.

These are the equations of motion on a rotating sheet of water, only that the angular velocity ω is replaced by $\omega \cos\theta$ or $\omega \sin\varphi$, where φ is the latitude.

If a water-particle is left to move under the influence of inertia, and the deflecting force of the earth's rotation, it will move in an orbit which is nearly a circle whose radius is $\frac{V}{2\,\omega \sin\varphi}$, where V is the velocity of the particle. The period of one revolution is $\frac{12}{\sin\varphi}$ sidereal hours, and has been termed half a pendulum day. At the Pole itself it will be 12 sidereal hours.

The semi-diurnal tidal waves whose periods are of about 12 hours, will be strongly modified by the earth's rotation. The dominating lunar component M_2 whose speed is $1.4052 \cdot 10^{-4}$ per sec. has a period which is equal to 12 pendulum hours at the latitude of 74° 30′ N. At this critical latitude it will have a tendency to degenerate. The wave-length will increase and the velocity of propagation will tend to become infinite. In reality this will never occur because all oceans are limited.

If we apply the equations to a sheet of relatively small dimensions, we may treat $\cos\theta$ as contant, and the motion will take place according to the same laws as on a plane rotating sheet of water. This is what we shall assume throughout this paper.

If we put

$$\lambda = 2\,\omega \cos\theta = 2\,\omega \sin\varphi$$

the equations of motion may be written:

$$\frac{\partial u}{\partial t} - \lambda v = -g\frac{\partial \zeta}{\partial x}$$

$$\frac{\partial v}{\partial t} + \lambda u = -g\frac{\partial \zeta}{\partial y}$$

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial (hu)}{\partial x} - \frac{\partial (hv)}{\partial y}.$$

The terms $-\lambda v$ and λu represent the deflecting force of the earth's rotation.

As the depth and limitation of the oceans make it impossible to obtain an exact solution, we shall have to simplify the problem considerably. We shall, therefore, consider only the propagation of waves in an infinite rotating layer or an infinitely long canal of uniform depth. This theory can be applied to conditions in the immediate vicinity of the Pole, but approximately the solution will apply also to limited areas of the Polar Ocean, if the double angular velocity $2\,\omega$ is replaced by $\lambda = 2\,\omega \sin\varphi$.

The considerations of progressive waves will be justified when we have treated the influence of viscossity upon the tidal waves. It can then be shown that the reflected waves in our case are so insignificant that they can be left out of consideration.

2. Tidal Motion in Canals.

The equations of motion have been integrated in a few cases. The propagation of free waves in an infinitely long canal has been investigated by Lord Kelvin. The solution is:

$$\zeta = C\,e^{-\frac{\lambda y}{c}} \cos\sigma\left(t - \frac{x}{c}\right),$$

$$u = \sqrt{\frac{g}{h}}\,C\,e^{-\frac{\lambda y}{c}} \cos\sigma\left(t - \frac{x}{c}\right), \qquad (1)$$

$$v = 0.$$

The wave-velocity is not affected by the rotation, but the amplitude is no longer the same if we go from one side of the canal to the other, being least on the side which is forward in respect of the rotation.

Another solution is found by Poincaré.[1])

To get this solution we may put more generally

$$\begin{aligned} u &= \mathrm{U}(y)\cos\sigma\left(t-\frac{x}{\varkappa}\right) \\ v &= \mathrm{V}(y)\sin\sigma\left(t-\frac{x}{\varkappa}\right) \\ \zeta &= \mathrm{Z}(y)\cos\sigma\left(t-\frac{x}{\varkappa}\right) \end{aligned} \tag{2}$$

and the condition to be satisfied at the boundaries $y = \pm b$ is $v = 0$.

Introducing the above expressions in the equations of motion, we get:

$$-\sigma\mathrm{U}\sin\sigma\left(t-\frac{x}{\varkappa}\right)-\lambda\mathrm{V}\sin\sigma\left(t-\frac{x}{\varkappa}\right)=-g\,\mathrm{Z}\,\frac{\sigma}{\varkappa}\sin\sigma\left(t-\frac{x}{\varkappa}\right)$$

$$\sigma\mathrm{V}\cos\sigma\left(t-\frac{x}{\varkappa}\right)+\lambda\mathrm{U}\cos\sigma\left(t-\frac{x}{\varkappa}\right)=-g\,\frac{d\,\mathrm{Z}}{d\,y}\cos\sigma\left(t-\frac{x}{\varkappa}\right),$$

and the equation of continuity gives:

$$-\sigma\mathrm{Z}\sin\sigma\left(t-\frac{x}{\varkappa}\right)=-h\left(\frac{\sigma}{\varkappa}\mathrm{U}\sin\sigma\left(t-\frac{x}{\varkappa}\right)+\frac{d\mathrm{V}}{dy}\sin\sigma\left(t-\frac{x}{\varkappa}\right)\right).$$

If we drop the trigonometrical factors, we get the equations:

$$\begin{aligned} \sigma\mathrm{U}+\lambda\mathrm{V} &= g\,\frac{\sigma}{\varkappa}\,\mathrm{Z} \\ \sigma\mathrm{V}+\lambda\mathrm{U} &= -g\,\frac{d\,\mathrm{Z}}{d\,y} \\ \sigma\mathrm{Z} &= h\,\frac{\sigma}{\varkappa}\,\mathrm{U}+h\,\frac{d\mathrm{V}}{dy}. \end{aligned} \tag{3}$$

Eliminating in turn U and V, we get

$$\begin{aligned} (\sigma^2-\lambda^2)\,\mathrm{U} &= g\left(\frac{\sigma^2}{\varkappa}\,\mathrm{Z}+\lambda\,\frac{d\,\mathrm{Z}}{dy}\right) \\ (\sigma^2-\lambda^2)\,\mathrm{V} &= -g\left(\frac{\sigma\lambda}{\varkappa}\,\mathrm{Z}+\sigma\,\frac{d\,\mathrm{Z}}{dy}\right). \end{aligned} \tag{4}$$

Carrying these values of U and V into the equation of continuity, it takes the form:

$$\frac{d^2\,\mathrm{Z}}{dy^2}+\left(\frac{\sigma^2-\lambda^2}{c^2}-\frac{\sigma^2}{\varkappa^2}\right)\mathrm{Z}=0. \tag{5}$$

If we put

$$\frac{\sigma^2-\lambda^2}{c^2}-\frac{\sigma^2}{\varkappa^2}=\frac{\pi^2}{4b^2}, \tag{6}$$

[1]) Poincaré: Leçons de mécanique celeste, t. III, p. 126.

we have the integral

$$Z = A \cos \frac{\pi y}{2b} + B \sin \frac{\pi y}{2b}. \tag{7}$$

Introducing this value of Z, we find:

$$(\sigma^2 - \lambda^2)\, V = -g \frac{\sigma\lambda}{\varkappa}\left(A \cos \frac{\pi y}{2b} + B \sin \frac{\pi y}{2b}\right) - g \frac{\sigma\pi}{2b}\left(-A \sin \frac{\pi y}{2b} + B \cos \frac{\pi y}{2b}\right)$$

$$= -g\left(\frac{\sigma\lambda}{\varkappa} A + \frac{\sigma\pi}{2b} B\right) \cos \frac{\pi y}{2b} - g\left(\frac{\sigma\lambda}{\varkappa} B - \frac{\sigma\pi}{2b} A\right) \sin \frac{\pi y}{2b}. \tag{8}$$

The conditions at the boundaries are satisfied if we put

$$\frac{\lambda}{\varkappa} B = \frac{\pi}{2b} A, \qquad B = \frac{\pi\varkappa}{2b\lambda} A.$$

We then find for V the expression:

$$V = -\frac{g\lambda\varkappa}{\sigma(\sigma^2 - \lambda^2)}\left(\frac{\sigma^2}{\varkappa^2} + \frac{\pi^2\sigma^2}{4b^2\lambda^2}\right) A \cos \frac{\pi y}{2b} \tag{9}$$

or if we replace $\frac{\sigma^2}{\varkappa^2}$ by $\frac{\sigma^2 - \lambda^2}{c^2} - \frac{\pi^2}{4b^2}$,

$$V = -A \frac{\varkappa}{h} \frac{\lambda}{\sigma}\left(1 + \frac{\pi^2 c^2}{4b^2\lambda^2}\right) \cos \frac{\pi y}{2b}. \tag{9^1}$$

For Z we then have:

$$Z = A\left(\cos \frac{\pi y}{2b} + \frac{\pi\varkappa}{2b\lambda} \sin \frac{\pi y}{2b}\right),$$

and for U we obtain:

$$U = \frac{g}{\sigma^2 - \lambda^2} \frac{\sigma^2}{\varkappa} A\left(\cos \frac{\pi y}{2b} + \frac{\pi\varkappa}{2b\lambda} \sin \frac{\pi y}{2b}\right) + \frac{g\lambda}{\sigma^2 - \lambda^2} \frac{\pi}{2b} A\left(-\sin \frac{\pi y}{2b} + \frac{\pi\varkappa}{2b\lambda} \cos \frac{\pi y}{2b}\right),$$

$$\text{or} \quad U = A\left(\frac{\varkappa}{h} \cos \frac{\pi y}{2b} + \frac{\pi g}{2b\lambda} \sin \frac{\pi y}{2b}\right). \tag{10}$$

The complete solution is, therefore:

$$\begin{aligned} u &= A\left(\frac{\varkappa}{h} \cos \frac{\pi y}{2b} + \frac{\pi g}{2b\lambda} \sin \frac{\pi y}{2b}\right) \cos \sigma\left(t - \frac{x}{\varkappa}\right) \\ v &= A \frac{\varkappa}{h} \frac{\lambda}{\sigma}\left(1 + \frac{\pi^2 c^2}{4b^2\lambda^2}\right) \cos \frac{\pi y}{2b} \sin \sigma\left(t - \frac{x}{\varkappa}\right) \\ \zeta &= A\left(\cos \frac{\pi y}{2b} + \frac{\pi\varkappa}{2b\lambda} \sin \frac{\pi y}{2b}\right) \cos \sigma\left(t - \frac{x}{\varkappa}\right). \end{aligned} \tag{11}$$

This wave is only possible if the following condition is fulfilled:

$$\sigma^2 > \lambda^2 + \frac{\pi^2 c^2}{4b^2}$$

and therefore forcibly

$$\sigma > \lambda, \qquad \sigma > \frac{\pi c}{2b}.$$

This means that the period must be shorter than half a pendulum day, and also shorter than the period of the free oscillation of a basin whose length is equal to the breath of the canal. This condition limits the application of the solution, and in a narrow canal the solution of Kelvin is the only one possible.

The wave can not be regularly reflected for if we replace x by $-x$, the factor of u which contains y is altered and it is impossible to find a section where u vanishes for all values of y.

3. A Solution with Vertical Motion.

If we allow vertical motion, we may get a solution which is valid in the case $\sigma < \lambda$. The equations of motion are:

$$\begin{aligned} \frac{\partial u}{\partial t} - \lambda v &= -\frac{1}{\varrho}\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t} + \lambda u &= -\frac{1}{\varrho}\frac{\partial p}{\partial y} \\ \frac{\partial w}{\partial t} &= -\frac{1}{\varrho}\frac{\partial p}{\partial z} - g \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 . \end{aligned} \tag{1}$$

We assume $p = p_0 + g\varrho\,(h - z) + g\varrho \mathrm{P} e^{i\sigma\left(t - \frac{x}{z}\right)}$

and $u,\ v,\ w \sim e^{i\sigma\left(t - \frac{x}{z}\right)}$.

We then get:

$$\begin{aligned} i\sigma u - \lambda v &= g\frac{i\sigma}{z}\mathrm{P} \\ i\sigma v + \lambda u &= -g\frac{\partial \mathrm{P}}{\partial y} \\ i\sigma w &= -g\frac{\partial \mathrm{P}}{\partial z} \\ \frac{i\sigma}{z}u &= \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} . \end{aligned} \tag{2}$$

Eliminating in turn u and v, we get:

$$\begin{aligned} (\sigma^2 - \lambda^2)\,u &= g\left(\frac{\sigma^2}{z}\mathrm{P} + \lambda\frac{\partial \mathrm{P}}{\partial y}\right) \\ (\sigma^2 - \lambda^2)\,v &= gi\sigma\left(\frac{\lambda}{z}\mathrm{P} + \frac{\partial \mathrm{P}}{\partial y}\right) \\ w &= -\frac{g}{i\sigma}\frac{\partial \mathrm{P}}{\partial z} . \end{aligned} \tag{3}$$

If we introduce these expressions in the equation of continuity, we get:

$$\frac{i\sigma}{\varkappa}\left(\frac{\sigma^2}{\varkappa}\,\mathrm{P}+\lambda\frac{\partial \mathrm{P}}{\partial y}\right)=i\sigma\left(\frac{\lambda}{\varkappa}\frac{\partial \mathrm{P}}{\partial y}+\frac{\partial^2 \mathrm{P}}{\partial y^2}\right)-\frac{\sigma^2-\lambda^2}{i\sigma}\frac{\partial^2 \mathrm{P}}{\partial z^2},$$

or

$$\frac{\sigma^2}{\varkappa^2}\,\mathrm{P}=\frac{\partial^2 \mathrm{P}}{\partial y^2}-\frac{\lambda^2-\sigma^2}{\sigma^2}\frac{\partial^2 \mathrm{P}}{\partial z^2}. \tag{4}$$

We assume

$$\frac{\partial^2 \mathrm{P}}{\partial y^2}=-\beta^2\mathrm{P}, \qquad \beta=\frac{\pi}{2b},$$

and get

$$\frac{d^2\mathrm{P}}{dz^2}(s^2-1)+\left(\beta^2+\frac{\sigma^2}{\varkappa^2}\right)\mathrm{P}=0, \qquad s=\frac{\lambda}{\sigma}, \tag{5}$$

or if we put $(s^2-1)\gamma^2=\beta^2+\frac{\sigma^2}{\varkappa^2}$, we have a solution

$$\mathrm{P}\sim\cos\gamma z.$$

This satisfies the condition $\frac{\partial \mathrm{P}}{\partial z}=0, \quad z=0.$

As function of y we take

$$\mathrm{P}=(\mathrm{A}\cos\beta y+\mathrm{B}\sin\beta y)\cos\gamma z. \tag{6}$$

The condition $v=0$ for $y=\pm b$ gives

$$\frac{\lambda}{\varkappa}\,\mathrm{P}+\frac{\partial \mathrm{P}}{\partial y}=0, \quad y=\pm b,$$

or

$$\left(\frac{\lambda}{\varkappa}\,\mathrm{A}+\beta\mathrm{B}\right)\cos\beta y+\left(\frac{\lambda}{\varkappa}\,\mathrm{B}-\beta\mathrm{A}\right)\sin\beta y$$

must be zero for $y=\pm b$. We, therefore, take

$$\mathrm{A}=\frac{\lambda}{\varkappa}\,\mathrm{A}_0, \quad \mathrm{B}=\beta\,\mathrm{A}_0,$$

and lastly

$$\mathrm{P}=\mathrm{A}_0\left(\frac{\lambda}{\varkappa}\cos\beta y+\beta\sin\beta y\right)\cos\gamma z=\psi(y)\cos\gamma z. \tag{7}$$

The condition $p=p_0$ for $z=h+\zeta$ gives

$$\zeta=\psi(y)\cos\gamma(h+\zeta).$$

The value of w for $z=h$ must be equal to $\frac{\partial\zeta}{\partial t}$, we, therefore, have

$$i\sigma\zeta=w_h=\frac{g\gamma}{i\sigma}\psi(y)\sin\gamma(h+\zeta),$$

or if we assume that ζ is small in comparison to h, we obtain

$$\frac{g\gamma}{\sigma^2}\sin\gamma h=-\cos\gamma h$$

$$\tan\gamma h=-\frac{\sigma^2 h^2}{gh\gamma h}. \tag{8}$$

This equation has an infinity of real roots greater than $\frac{\pi}{2}$, and as $\frac{\sigma^2 h^2}{c^2 \gamma h}$ is a small quantity, we see that the smallest root is aproximately equal to π, or $\gamma = \frac{\pi}{h}$.

This gives

$$\frac{\sigma^2}{\varkappa^2} = \frac{\pi^2}{h^2}(s^2 - 1) - \frac{\pi^2}{4b^2} \text{ or as } \frac{\sigma}{\varkappa} = \frac{2\pi}{T\varkappa} = \frac{2\pi}{L} \tag{9}$$

where L is the wave-length:

$$\left(\frac{2}{L}\right)^2 = \frac{s^2 - 1}{h^2} - \frac{1}{4b^2}. \tag{10}$$

If b is very great in comparison to h, we have approximately:

$$L = \frac{2h}{\sqrt{s^2 - 1}}. \tag{11}$$

We see that the wave-length is generally small and thus the propagation very slow.

Introducing the value of P in the expressions for u, v and w, we obtain:

$$u = A_0 \frac{g}{\varkappa}\left(\beta \sin \beta y - \frac{\lambda \gamma^2 \varkappa}{\sigma^2} \cos \beta y\right) \cos \gamma z,$$

$$v = -\frac{i\sigma}{\lambda^2 - \sigma^2} g A_0 \left(\frac{\lambda^2}{\varkappa^2} + \beta^2\right) \cos \beta y \cos \gamma z,$$

$$w = \frac{g\gamma}{i\sigma} A_0 \left(\frac{\lambda}{\varkappa} \cos \beta y + \beta \sin \beta y\right) \sin \gamma z,$$

or if we restore the timefactors:

$$\begin{aligned} u &= A_0 \frac{g}{\varkappa}\left(\beta \sin \beta y - \frac{\lambda \gamma^2 \varkappa}{\sigma^2} \cos \beta y\right) \cos \gamma z \cos \sigma\left(t - \frac{x}{\varkappa}\right) \\ v &= \frac{A_0 g}{\sigma(s^2 - 1)}\left(\frac{\lambda^2}{\varkappa^2} + \beta^2\right) \cos \beta y \cos \gamma z \sin \sigma\left(t - \frac{x}{\varkappa}\right) \\ w &= \frac{A_0 g \gamma}{\sigma}\left(\frac{\lambda}{\varkappa} \cos \beta y + \beta \sin \beta y\right) \sin \gamma z \sin \sigma\left(t - \frac{x}{\varkappa}\right). \end{aligned} \tag{12}$$

The surface elevation is expressed by:

$$\zeta = A_0 \left(\frac{\lambda}{\varkappa} \cos \beta y + \beta \sin \beta y\right) \cos \gamma h \cos \sigma\left(t - \frac{x}{\varkappa}\right). \tag{13}$$

If we replace γ by its approximate value $\frac{\pi}{h}$, we get

$$\begin{aligned} u &= A_0 \frac{g}{\varkappa}\left(\frac{\pi}{2b} \sin \frac{\pi y}{2b} - \frac{\lambda \pi^2 \varkappa}{h^2 \sigma^2} \cos \frac{\pi y}{2b}\right) \cos \frac{\pi z}{h} \cos \sigma\left(t - \frac{x}{\varkappa}\right) \\ v &= \frac{A_0 g}{\sigma(s^2 - 1)}\left(\frac{\lambda^2}{\varkappa^2} + \frac{\pi^2}{4b^2}\right) \cos \frac{\pi y}{2b} \cos \frac{\pi z}{h} \sin \sigma\left(t - \frac{x}{\varkappa}\right) \end{aligned} \tag{14}$$

$$w = A_0 \frac{g\pi}{h}\left(\frac{\lambda}{\varkappa} \cos \frac{\pi y}{2b} + \frac{\pi}{2b} \sin \frac{\pi y}{2b}\right) \sin \frac{\pi z}{h} \sin \sigma\left(t - \frac{x}{\varkappa}\right) \tag{15}$$

$$\zeta = -A_0 \left(\frac{\lambda}{\varkappa} \cos \frac{\pi y}{2b} + \frac{\pi}{2b} \sin \frac{\pi y}{2b}\right) \cos \sigma\left(t - \frac{x}{\varkappa}\right).$$

We note that u, v and ζ change their signs if we pass from one side of the canal to the other. We find:

$$u = 0 \quad \text{for}$$

$$\tan \frac{\pi y}{2b} = \frac{\lambda \pi 2b\varkappa}{h^2 \sigma^2} = s \frac{b\,\mathrm{L}}{h^2}.$$

This is greater than 1 and therefore $y > \frac{b}{2}$. Consequently the sign of u is changed in the left fourth of the canal. u has a maximum for

$$\tan \frac{\pi y}{2b} = -\frac{h^2}{s\,b\,\mathrm{L}}.$$

This is near the middle of the canal and on the right side.

w and ζ are zero for

$$\tan \frac{\pi y}{2b} = -\frac{2b\lambda}{\pi\varkappa} = -s\frac{4b}{\mathrm{L}}.$$

This is near the right side.

There is high water on the right side when there is low water on the left and vice versa. However I do not think that this solution may have any application to tidal phenomena.

4. Canal with Frictional Resistance.

In connection with these examples of wave-motion in canals, we may also see how the waves are modified if there is a resistance. For simplicity we only modify the equations by the addition of a frictional term, which is proportional to the velocity. The equations of motion may then be written

$$\begin{aligned} \frac{\partial u}{\partial t} + \eta u &= -g\frac{\partial \zeta}{\partial x} \\ \lambda u &= -g\frac{\partial \zeta}{\partial y}. \end{aligned} \tag{1}$$

The equation of continuity is

$$\frac{\partial \zeta}{\partial t} = -h\frac{\partial u}{\partial x}. \tag{2}$$

That is to say, we have only added the frictional term ηu.

We assume

$$u, \zeta \sim e^{i\sigma t - \gamma x - my},$$

and get:

$$\begin{aligned} (i\sigma + \eta)\, u &= g\gamma \mathrm{C} \\ \lambda u &= gm\mathrm{C} \\ i\sigma \mathrm{C} &= h\gamma u. \end{aligned} \tag{3}$$

From these equations we find:

$$\gamma^2 = -\frac{\sigma^2}{c^2} + \frac{i\sigma\eta}{c^2}, \qquad c^2 = gh,$$
$$m^2 = \left(\frac{\sigma^2}{c^2} + \frac{i\sigma\eta}{c^2}\right)\frac{\lambda^2}{\sigma^2+\eta^2}, \tag{4}$$

or if we put

$$\gamma = \mu + \frac{i\sigma}{\varkappa}, \qquad \frac{\lambda^2}{\sigma^2+\eta^2} = \alpha^2,$$

we get

$$\frac{\sigma^2}{\varkappa^2} - \mu^2 = \frac{\sigma^2}{c^2}, \qquad 2\mu\frac{\sigma}{\varkappa} = \frac{\sigma\eta}{c^2}, \qquad m = \left(\frac{\sigma}{\varkappa} + i\mu\right)\alpha.$$

We put

$$\mu = \frac{\sigma}{\varkappa}p, \qquad \varkappa = \frac{c}{\vartheta}.$$

For p and ϑ we then have the equations:

$$\vartheta^2(1-p^2) = 1, \qquad 2p\vartheta^2 = \frac{\eta}{\sigma} \tag{5}$$

which give

$$\frac{2p}{1-p^2} = \frac{\sigma}{\eta}, \tag{6}$$

or if we put $p = \tan\gamma$, we get

$$\tan 2\gamma = \frac{\sigma}{\eta}, \quad p = \frac{\sqrt{\sigma^2+\eta^2}-\sigma}{\eta}$$
$$\vartheta^2 = \frac{\sqrt{\sigma^2+\eta^2}+\sigma}{2\sigma}.$$

For u we then obtain:

$$u = \frac{g\,m\,C}{\lambda}\,e^{i\sigma\left(t-\frac{x+\alpha py}{\varkappa}\right)-\frac{\sigma}{\varkappa}(px+\alpha y)}$$
$$= \frac{g}{\varkappa}\frac{(1+ip)\,\alpha\sigma}{\lambda}\,Ce^{i\sigma\left(t-\frac{x+\alpha py}{\varkappa}\right)-\frac{\sigma}{\varkappa}(px+\alpha y)},$$

or if we put

$$1+ip = \sqrt{1+p^2}\,e^{i\gamma}, \qquad \text{we find}$$

$$u = \frac{g}{\varkappa}\frac{\sigma\sqrt{1+p^2}}{\sqrt{\sigma^2+\eta^2}}\,C\cos\left[\sigma\left(t-\frac{x+\alpha py}{\varkappa}\right)+\gamma\right]e^{-\frac{\sigma}{\varkappa}(px+\alpha y)},$$

or as

$$\frac{\sigma}{\varkappa}\sqrt{1+p^2} = \frac{1}{c}\sqrt{\sigma\sqrt{\sigma^2+\eta^2}}:$$

$$u = \sqrt{\frac{g}{h}}\,C\sqrt{\frac{\sigma}{\sqrt{\sigma^2+\eta^2}}}\cos\left[\sigma\left(t-\frac{x+\alpha py}{\varkappa}\right)+\gamma\right]e^{-\frac{\sigma}{\varkappa}(px+\alpha y)}$$
$$\zeta = C\cos\sigma\left(t-\frac{x+\alpha py}{\varkappa}\right)e^{-\frac{\sigma}{\varkappa}(px+\alpha y)}. \tag{7}$$

The time of high water is

$$t = \frac{x + \alpha p y}{\varkappa} . \tag{8}$$

The phase-difference between surface-elevation and current is $-\gamma$, so that maximum velocity occurs before high water. The cotidal lines are expressed by the equation:

$$x + \alpha p y = \text{const.}, \tag{9}$$

they form with the y-axis an angle whose tangent is αp. ζ is constant along the lines whose equation is

$$p x + \alpha y = \text{const.} \tag{10}$$

The wave apparently proceeds in a direction which forms an angle with the x-axis, whose tangent is αp. One should, therefore, think that the wave must be reflected at the side of the canal, but as the velocity has no component in this direction no reflection takes place.

5. Tidal Motion in Extensive Areas.

Another case in which the equations have been integrated applies to the propagation of free waves on an unlimited rotating disc. As this case is treated in detail in the pulication of H. U. Sverdrup, we may only make a short recapitulation of the results. The solution is in this case:

$$\begin{aligned} u &= \sqrt{\frac{g}{h}} \frac{C}{\sqrt{1 - s^2}} \cos \sigma \left(t - \frac{x}{\varkappa} \right) \\ v &= - \sqrt{\frac{g}{h}} \frac{s C}{\sqrt{1 - s^2}} \sin \sigma \left(t - \frac{x}{\varkappa} \right) \\ \zeta &= C \cos \sigma \left(t - \frac{x}{\varkappa} \right), \quad \text{where } s = \frac{\lambda}{\sigma} . \end{aligned} \tag{1}$$

The velocity of propagation is

$$\varkappa = \frac{c}{\sqrt{1 - s^2}}, \quad c^2 = g h \tag{2}$$

and for $\sigma = \lambda$ the velocity will be infinitely great. The wave is then degenerated and the motion circular, $\zeta = 0$. The solution is, therefore, only valid as long as $\sigma > \lambda$, and the result can not be applied to the actual tidal conditions in the Arctic Ocean, as only the higher harmonics fulfil the condition $\sigma > \lambda$.

For instance, the lunar wave M_2 has the speed $\sigma = 1.4052 \cdot 10^{-4}$ and the double angular velocity of the earth is $2\ \omega = 1.45842 \cdot 10^{-4}$.

Undoubtedly there exist free waves in the Arctic Ocean owing to the fact that the Ocean is limited and the deflecting force is balanced by a transversal slope of the wave crests. This effect of the boundaries may to a certain degree be taken into account in the solution. It is possible to build up a solution which will be valid for a limited area, not too near the coast.

Let us consider a somewhat broad channel. If the space near the coast is left out of consideration, the motion of the particles may be elliptic, but the coast will prevent the full development of the rotary currents. The deflecting force of the earth's rotation then will be partially balanced by a transversal slope of the wave-crest.

We may assume

$$\begin{aligned} \zeta &= C e^{-my + i\sigma\left(t - \frac{x}{\varkappa}\right)} \\ u &= A e^{-my + i\sigma\left(t - \frac{x}{\varkappa}\right)} \\ v &= B e^{-my + i\sigma\left(t - \frac{x}{\varkappa}\right)} . \end{aligned} \tag{3}$$

If we introduce these values in the equations of motion we get:

$$i\sigma A - \lambda B = g \frac{i\sigma}{\varkappa} C$$

$$i\sigma B + \lambda A = gmC ,$$

$$A = -gC \frac{m\lambda - \frac{\sigma^2}{\varkappa}}{\sigma^2 - \lambda^2}, \qquad B = gC \frac{i\sigma\left(\frac{\lambda}{\varkappa} - m\right)}{\sigma^2 - \lambda^2} .$$

The equation of continuity is

$$i\sigma C = h\left(\frac{i\sigma}{\varkappa} A + m B\right), \quad \text{or } 1 = -\frac{gh}{\sigma^2 - \lambda^2}\left(m^2 - \frac{\sigma^2}{\varkappa^2}\right). \tag{4}$$

We may now assume $m = \frac{\lambda}{c}\sqrt{1-\theta^2}$ where θ is yet undetermined.

This gives

$$\frac{\sigma^2}{\varkappa^2} = \frac{\sigma^2}{c^2}\left(1 - \frac{\lambda^2\theta^2}{\sigma^2}\right) = \frac{\sigma^2}{c^2}\left(1 - s^2\theta^2\right), \quad \varkappa = \frac{c}{\sqrt{1 - s^2\theta^2}} . \tag{5}$$

Introducing these values in A and B we get

$$A = \sqrt{\frac{g}{h}}\, C \frac{\sqrt{1-s^2\theta^2} - s^2\sqrt{1-\theta^2}}{1-s^2}, \qquad B = i\sqrt{\frac{g}{h}}\, Cs \frac{\sqrt{1-s^2\theta^2} - \sqrt{1-\theta^2}}{1-s^2} . \tag{6}$$

In real form the solution will be

$$\begin{aligned} u &= \sqrt{\frac{g}{h}}\, C \frac{\sqrt{1-s^2\theta^2} - s^2\sqrt{1-\theta^2}}{1-s^2} \cos\sigma\left(t - \frac{x}{\varkappa}\right) e^{-my} \\ v &= -\sqrt{\frac{g}{h}}\, Cs \frac{\sqrt{1-s^2\theta^2} - \sqrt{1-\theta^2}}{1-s^2} \sin\sigma\left(t - \frac{x}{\varkappa}\right) e^{-my} \\ \zeta &= C \cos\sigma\left(t - \frac{x}{\varkappa}\right) e^{-my} \end{aligned} \tag{7}$$

where $s = \frac{\lambda}{\sigma}$.

This solution will still be valid if we put $s = 1$. The coefficients of u and v will appear in the form $\frac{0}{0}$, but the values are easily found to be

$$\sqrt{\frac{g}{h}}\, C\, \frac{2-\theta^2}{2\sqrt{1-\theta^2}}, \qquad \sqrt{\frac{g}{h}}\, C\, \frac{\theta^2}{2\sqrt{1-\theta^2}}. \tag{8}$$

The motion is elliptic, and the ratio between the minor and the major axis of the ellipse is

$$\alpha = \frac{s\left(\sqrt{1-s^2\theta^2} - \sqrt{1-\theta^2}\right)}{\sqrt{1-s^2\theta^2} - s^2\sqrt{1-\theta^2}}.$$

We then get

$$\sqrt{1-s^2\theta^2} = \frac{1-\alpha s}{\sqrt{1-\alpha^2}}, \qquad \sqrt{1-\theta^2} = \frac{s-\alpha}{s\sqrt{1-\alpha^2}}.$$

If we put

$$\sqrt{\frac{g}{h}}\, C\, \frac{\sqrt{1-s^2\theta^2} - s^2\sqrt{1-\theta^2}}{1-s^2} = A.$$

we get

$$C = \sqrt{\frac{h}{g}}\, \sqrt{1-\alpha^2}\, A, \qquad B = i\alpha A$$

and

$$\begin{aligned} u &= A \cos \sigma \left(t - \frac{x}{\varkappa}\right) e^{-my} \\ v &= -\alpha A \sin \sigma \left(t - \frac{x}{\varkappa}\right) e^{-my} \\ \zeta &= \sqrt{\frac{h}{g}}\, \sqrt{1-\alpha^2}\, A \cos \sigma \left(t - \frac{x}{\varkappa}\right) e^{-my}, \\ \varkappa &= c\, \frac{\sqrt{1-\alpha^2}}{1-\alpha s}, \qquad m = \frac{\lambda - \alpha\sigma}{c\sqrt{1-\alpha^2}}. \end{aligned} \tag{9}$$

This solution is also given by Sverdrup[1]) though with somewhat different notations.

Now the discussion of the motion is simple. If we put $\alpha = 0$, we have the case of a uniform canal. If we put $\alpha = s$, we have the case of the unlimited disc. We see that if we suppose A to have a constant value, ζ will decrease as the motion becomes more elliptic, and if we allow s to be 1, $\alpha = 1$, the wave will degenerate. We then have $|u| = |v|$. The whole mass of water will move in the circle of inertia. This fact is important when dealing with the tidal waves in the Arctic Ocean. The waves enter the polar basin through the narrow channel between Greenland and Spitzbergen (Svalbard). In this channel the motion will be practically alternating. As the channel widens, the tidal currents must become rotating, and the vertical elevation of the surface must decrease. We must, therefore, expect that the amplitude of the wave decreases as we pass across the polar basin and that the elevation becomes small in comparison with the tidal currents. A good illustration of this is given by the tide at Point

[1]) L. C. pag. 24.

Barrow, which has also been pointed out by Sverdrup. The currents seem to indicate that the flood comes from the west as Harris supposed, but in reality it may only be an effect of the rotary character of the current. The small elevation of the surface indicates that the wave is nearly degenerated, and that the motion is almost circular.

The diminution of the tidal wave is thus explained. The same reasoning might be applied to the propagation of the wave on the continental shelf north of Siberia, and would to a certain extent explain the diminution of the tidal wave as observed at Ayon Island. But as the depth of sea is small, one must expect that the resistance will strongly modify the tidal motion and it is therefore advisable first to investigate the effect of frictional influence upon the waves.

6. Influence of Frictional Resistance on Tidal Waves.

The motion is supposed to take place in a very wide channel, and the motion near the shores shall be left out of account. The water-particles may then move in elliptic orbits, but the ratio of the minor and major axes of the ellipse is smaller than in the case of an unlimited ocean. The deflecting force of the earth's rotation is partly balanced by the slope of the wave-crests. It is true that the wave in this case will not proceed without change of type, but for limited areas the solution will be valid. As the conditions are never such that the waves propagate without a change, the above solution might apply better to the observed conditions than the solution which is valid for a free wave in an unlimited layer on a rotating disc.

The restrictions under which the equations of motion are valid, have been treated by Sverdrup, and I may, therefore, only briefly touch some points.

The x-axis is parallell to the direction of propagation. The z-axis points vertically upwards and the y-axis is perpendicular to the xz plane, and the positive direction is such that the x-axis by a rotation of 90° in the direction in which the disc rotates will coïncide with the y-axis. The depth will be supposed to be constant and the ordinate of the free undisturbed surface will be denoted by h. The ordinate of the disturbed surface at the time t shall be $h + \zeta$.

The double angular velocity of the rotating disc will be denoted by λ. When applying the results to the rotating earth, λ is replaced by $2\omega\sin\varphi$, ω being the angular velocity of the earth and φ the latitude. The eddy viscosity ν is supposed to be constant.

The equations of motion are then:

$$\begin{aligned} \frac{\partial u}{\partial t} - \lambda v &= -g\frac{\partial \zeta}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} \\ \frac{\partial v}{\partial t} + \lambda u &= -g\frac{\partial \zeta}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}. \end{aligned} \tag{1}$$

The equation of continuity is:

$$\frac{\partial \zeta}{\partial t} = -\int_0^{h+\zeta}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz, \tag{2}$$

but as ζ is supposed to be only a small quantity in comparison to h, this equation may be written:

$$\frac{\partial \zeta}{\partial t} = -\int_0^{h}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz. \tag{2^1}$$

We may now assume that u, v and ζ can be expressed in the following form:

$$\begin{aligned} u &= \mathrm{U}\, e^{-\mu x - my + i\sigma\left(t - \frac{x}{z}\right)} \\ v &= \mathrm{V}\, e^{-\mu x - my + i\sigma\left(t - \frac{x}{z}\right)} \\ \zeta &= \mathrm{C}\, e^{-\mu x - my + i\sigma\left(t - \frac{x}{z}\right)}, \end{aligned} \tag{3}$$

where U and V are functions of z only, while C is constant. At the end of the calculation we reject the imaginary parts and retain the real parts only. We obtain:

$$\begin{aligned} i\sigma \mathrm{U} - \lambda \mathrm{V} &= g\left(\mu + \frac{i\sigma}{z}\right) \mathrm{C} + \nu \frac{d^2\mathrm{U}}{dz^2} \\ i\sigma \mathrm{V} + \lambda \mathrm{U} &= gm\mathrm{C} + \nu \frac{d^2\mathrm{V}}{dz^2}. \end{aligned} \tag{4}$$

These equations may be satisfied by putting

$$\mathrm{U} = \mathrm{U}_1 + \mathrm{U}_2, \quad \mathrm{V} = -i\,\mathrm{U}_1 + i\,\mathrm{U}_2. \tag{5}$$

U_1 and U_2 are then determined by the equations:

$$\begin{aligned} \frac{d^2\mathrm{U}_1}{dz^2} - i\frac{\sigma + \lambda}{\nu}\mathrm{U}_1 &= -i\frac{m - i\gamma}{2\nu} g\mathrm{C} \\ \frac{d^2\mathrm{U}_2}{dz^2} - i\frac{\sigma - \lambda}{\nu}\mathrm{U}_2 &= i\frac{m + i\gamma}{2\nu} g\mathrm{C}, \end{aligned} \tag{6}$$

where

$$\gamma = \mu + \frac{i\sigma}{z}.$$

If we put

$$\beta_1 = \sqrt{\frac{\sigma + \lambda}{2\nu}}, \quad \beta_2 = \sqrt{\frac{\sigma - \lambda}{2\nu}}, \tag{7}$$

we get

$$\begin{aligned} \mathrm{U}_1 &= \frac{m - i\gamma}{\sigma + \lambda}\frac{g\mathrm{C}}{2} + \mathrm{A}_1 e^{\beta_1(1+i)z} + \mathrm{A}_2 e^{-\beta_1(1+i)z} \\ \mathrm{U}_2 &= -\frac{m + i\gamma}{\sigma - \lambda}\frac{g\mathrm{C}}{2} + \mathrm{B}_1 e^{\beta_2(1+i)z} + \mathrm{B}_2 e^{-\beta_2(1+i)z}. \end{aligned} \tag{8}$$

The boundary conditions give equations for determination of the coefficients A_1, A_2, B_1 and B_2. As to the condition introduced at the surface we shall at first make a very general assumption.

In the arctic waters the surface will be covered by an ice-sheet during the greater part of the year. Especially in winter the ice will only to some degree take part in the horizontal movement, but will of course be lifted by the tidal waves. There must, therefore, be a tangential stress at the surface which is directed against the motion. As we neglect the vertical velocities, the tangential stress in the direction of the x-axis is $-\varrho\nu\frac{\partial u}{\partial z}$ and in the direction of the y-axis $-\varrho\nu\frac{\partial v}{\partial z}$ (ϱ density). This tangential stress must be some function of the velocity and also of the direction, and to some degree of the time, strictly speaking it should also be a function of the place. But as we only consider waves which have a definite wave-length, we must assume that the stress is independent of x and y, we may also assume that the ice-conditions do not alter in the course of one period and that we, therefore, may treat the stress as a function of the velocities only.

The simplest assumption is

$$\frac{\partial u}{\partial z} + fu = 0, \qquad \frac{\partial v}{\partial z} + fv = 0, \qquad z = h\,.$$

$f = 0$ then corresponds to the case of a free surface.

At the bottom we assume that there is no slipping and therefore $u = 0$ and $v = 0$. We obtain

$$\begin{aligned} U_1 &= \frac{m - i\gamma}{\sigma + \lambda}\,\frac{gC}{2}\left(1 - \frac{\varrho_1 \cosh \varrho_1 (h - z) + f(\sinh \varrho_1 (h - z) + \sinh \varrho_1 z)}{\varrho_1 \cosh \varrho_1 h + f \sinh \varrho_1 h}\right) \\ U_2 &= -\frac{m + i\gamma}{\sigma - \lambda}\,\frac{gC}{2}\left(1 - \frac{\varrho_2 \cosh \varrho_2 (h - z) + f(\sinh \varrho_2 (h - z) + \sinh \varrho_2 z)}{\varrho_2 \cosh \varrho_2 h + f \sinh \varrho_2 h}\right), \end{aligned} \tag{9}$$

$$\varrho = \beta(1 + i)\,.$$

In order to abreviate the writing we denote the hyperbolical functions in the parentheses by $\Phi_1(z)$ and $\Phi_2(z)$.

Further we put

$$\frac{1}{h}\int_0^h \Phi\, dz = A = \frac{1}{\varrho h}\,\frac{\varrho \sinh \varrho h + 2f(\cosh \varrho h - 1)}{\varrho \cosh \varrho h + f \sinh \varrho h}, \tag{10}$$

the equation of continuity then will be

$$i\sigma C = -\frac{m^2 + \gamma^2}{\sigma + \lambda}\,\frac{ghC}{2}(1 - A_1) - \frac{m^2 + \gamma^2}{\sigma - \lambda}\,\frac{ghC}{2}(1 - A_2)$$

or

$$\frac{\sigma^2 - \lambda^2}{c^2} = -(m^2 + \gamma^2)\left(\frac{\sigma - \lambda}{2\sigma}(1 - A_1) + \frac{\sigma + \lambda}{2\sigma}(1 - A_2)\right). \tag{11}$$

We shall not go further in details in the discussion of this equation, but threat the two cases $f = 0$ corresponding to free surface and $f = \infty$ corresponding to an ice-sheet which does not take part in the motion.

If we put $f = 0$ we get

$$\begin{aligned} U &= \frac{m - i\gamma}{\sigma + \lambda}\,\frac{gC}{2}\left(1 - \frac{\cosh \varrho_1 (h - z)}{\cosh \varrho_1 h}\right) - \frac{m + i\gamma}{\sigma - \lambda}\,\frac{gC}{2}\left(1 - \frac{\cosh \varrho_2 (h - z)}{\cosh \varrho_2 h}\right) \\ V &= -i\,\frac{m - i\gamma}{\sigma + \lambda}\,\frac{gC}{2}\left(1 - \frac{\cosh \varrho_1 (h - z)}{\cosh \varrho_1 h}\right) - i\,\frac{m + i\gamma}{\sigma - \lambda}\,\frac{gC}{2}\left(1 - \frac{\cosh \varrho_2 (h - z)}{\cosh \varrho_2 h}\right). \end{aligned} \tag{12}$$

The equation of continuity will be

$$-\frac{\sigma^2 - \lambda^2}{c^2} = (m^2 + \gamma^2)\left[\frac{1 - s}{2}\left(1 - \frac{\tanh \varrho_1 h}{\varrho_1 h}\right) + \frac{1 + s}{2}\left(1 - \frac{\tanh \sigma_2 h}{\varrho_2 h}\right)\right]. \tag{13}$$

This equation is deduced under the assumption $\sigma > \lambda$, and the equation shall first be solved under this supposition. Afterwards we shall see that this supposition may be dropped.

7. Solution of the Equation of the Continuity.

We may put

$$\varrho_1 = \sqrt{\frac{\sigma+\lambda}{2\nu}}\,(1+i) = \beta_1\,(1+i), \qquad \varrho_2 = \sqrt{\frac{\sigma-\lambda}{2\nu}}\,(1+i) = \beta_2\,(1+i), \tag{1}$$

and then we obtain:

$$\frac{\tanh \varrho_1 h}{\varrho_1 h} = \frac{\sinh 2\beta_1 h + \sin 2\beta_1 h}{2\beta_1 h\,(\cosh 2\beta_1 h + \cos 2\beta_1 h)} - i\,\frac{\sinh 2\beta_1 h - \sin 2\beta_1 h}{2\beta_1 h\,(\cosh 2\beta_1 h + \cos 2\beta_1 h)}$$
$$= a_1 - i b_1 . \tag{2}$$

For great values of $\beta_1 h$ we have approximately $a_1 = b_1 = \frac{1}{2\beta_1 h}$.

For $\frac{\tanh \varrho_2 h}{\varrho_2 h}$ we have the analogus expression $a_2 - i b_2$.

The equation of continuity may then be written:

$$-\frac{\sigma^2-\lambda^2}{c^2} = (m^2+\gamma^2)\left[1 - \frac{1-s}{2}(a_1 - i b_1) - \frac{1+s}{2}(a_2 - i b_2)\right], \tag{3}$$

or if we put:

$$\delta = \frac{1-s}{2}a_1 + \frac{1+s}{2}a_2, \qquad \varepsilon = \frac{1-s}{2}b_1 + \frac{1+s}{2}b_2, \tag{4}$$

and remember that $\gamma = \mu + \frac{i\sigma}{\varkappa}$:

$$-\frac{\sigma^2-\lambda^2}{c^2} = \left(m^2+\mu^2-\frac{\sigma^2}{\varkappa^2}\right)(1-\delta) - 2\mu\,\frac{\sigma}{\varkappa}\,\varepsilon$$
$$0 = \left(m^2+\mu^2-\frac{\sigma^2}{\varkappa^2}\right)\varepsilon + 2\mu\,\frac{\sigma}{\varkappa}(1-\delta). \tag{5}$$

From these two expressions we obtain:

$$m^2+\mu^2-\frac{\sigma^2}{\varkappa^2} = -\frac{\sigma^2-\lambda^2}{c^2}\,\frac{1-\delta}{(1-\delta)^2+\varepsilon^2} = -\frac{\sigma^2-\lambda^2}{c^2}\,k_1,$$
$$2\mu\,\frac{\sigma}{\varkappa} = \frac{\sigma^2-\lambda^2}{c^2}\,\frac{\varepsilon}{(1-\delta)^2+\varepsilon^2} = \frac{\sigma^2-\lambda^2}{c^2}\,k_2 . \tag{6}$$

When the eddy viscossity was left out of account we had

$$m = \frac{\lambda}{c}\sqrt{1-\theta^2}, \qquad \frac{\sigma}{\varkappa} = \frac{\sigma}{c}\sqrt{1-s^2\theta^2}.$$

In this case we also retain the indetermined form for m and put

$$m = \frac{\lambda}{c}\,\theta, \qquad \mu = \frac{\sigma}{\varkappa}\,p, \qquad \frac{\sigma}{\varkappa} = \frac{\sigma}{c}\,\vartheta .$$

The equations for p and ϑ then will be

$$(1-p^2)\,\vartheta^2 = (1-s^2)\,k_1 + s^2\theta^2, \qquad 2p\vartheta^2 = (1-s^2)\,k_2 . \tag{7}$$

From these two equations we get p and ϑ, and we may now build up the solution.

We find:

$$m - i\gamma = \frac{\lambda}{c}\,\theta + \frac{\sigma}{c}\,\vartheta\,(1 - ip),$$

$$m + i\gamma = \frac{\lambda}{c}\,\theta - \frac{\sigma}{c}\,\vartheta\,(1 - ip).$$

If we put

$$A = \frac{\vartheta + s\,\theta}{1 + s}, \quad A_1 = \frac{\vartheta\,p}{1 + s}, \quad B = \frac{\vartheta - s\,\theta}{1 - s}, \quad B_1 = \frac{\vartheta\,p}{1 - s}, \tag{8}$$

we get:

$$\begin{aligned} U &= \sqrt{\frac{g}{h}}\,\frac{C}{2}\left[(A - i\,A_1)\,(1 - \Phi_1) + (B - i\,B_1)\,(1 - \Phi_2)\right] \\ V &= \sqrt{\frac{g}{h}}\,\frac{C}{2}\left[-\,i\,(A - i\,A_1)\,(1 - \Phi_1) + i\,(B - i\,B_1)\,(1 - \Phi_2)\right]. \end{aligned} \tag{9}$$

Further we may put:

$$\Phi_1 = \varphi_1 - i\,\psi_1, \quad \Phi_2 = \varphi_2 - i\,\psi_2,$$

where

$$\begin{aligned} \varphi_1 &= \frac{\cosh\beta_1\,(2h - z)\cos\beta_1\,z + \cosh\beta_1\,z\cos\beta_1\,(2h - z)}{\cosh 2\beta_1\,h + \cos 2\beta_1\,h} \\ \psi_1 &= \frac{\sinh\beta_1\,(2h - z)\sin\beta_1\,z + \sinh\beta_1\,z\sin\beta_1\,(2h - z)}{\cosh 2\beta_1\,h + \cos 2\beta_1\,h}. \end{aligned} \tag{10}$$

For φ_2 and ψ_2 we get analogus expressions.

In real form we then obtain:

$$\begin{aligned} u &= \sqrt{\frac{g}{h}}\,\frac{C}{2}\left[M\cos\sigma\left(t - \frac{x}{\varkappa}\right) + N\sin\sigma\left(t - \frac{x}{\varkappa}\right)\right] e^{-\mu x - my} \\ v &= \sqrt{\frac{g}{h}}\,\frac{C}{2}\left[P\cos\sigma\left(t - \frac{x}{\varkappa}\right) + Q\sin\sigma\left(t - \frac{x}{\varkappa}\right)\right] e^{-\mu x - my}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} M &= A\,(1 - \varphi_1) + A_1\,\psi_1 + B\,(1 - \varphi_2) + B_1\,\psi_2, \\ N &= -\,A\,\psi_1 + A_1\,(1 - \varphi_1) - B\,\psi_2 + B_1\,(1 - \varphi_2), \\ P &= A\,\psi_1 - A_1\,(1 - \varphi_1) - B\,\psi_2 + B_1\,(1 - \varphi_2), \\ Q &= A\,(1 - \varphi_1) + A_1\,\psi_1 - B\,(1 - \varphi_2) - B_1\,\psi_2. \end{aligned} \tag{12}$$

The preceding solution applies so long as $\sigma > \lambda$. The case $\sigma = \lambda$ may easily be treated. We shall see that the equation of continuity gives a finite value for $\varkappa$.

At first we shall find the value of $\lim \frac{1 - \delta}{\sigma - \lambda}$ and $\lim \frac{\varepsilon}{\sigma - \lambda}$. We have

$$\frac{1 - \dfrac{\tanh\varrho_1 h}{\varrho_1 h}}{\sigma + \lambda} + \frac{1 - \dfrac{\tanh\varrho_2 h}{\varrho_2 h}}{\sigma - \lambda} = \frac{2\sigma\,(1 - \delta) + 2i\sigma\varepsilon}{\sigma^2 - \lambda^2}. \tag{13}$$

The first term remains finite for $\sigma = \lambda$ and the limit of this expression may be written

$$\frac{1 - a_1 + i\,b_1}{2\sigma}.$$

Developing $\tanh \varrho_2 h$ in series we have:

$$\tanh \varrho_2 h = \varrho_2 h - \frac{1}{3} \varrho_2{}^3 h^3 + \frac{2}{15} \varrho_2{}^5 h^5 \ldots\ldots, \tag{14}$$

and therefore

$$1 - \frac{\tanh \varrho_2 h}{\varrho_2 h} = \frac{1}{3} \varrho_2{}^2 h^2 - \frac{2}{15} \varrho_2{}^4 h^4 + \ldots\ldots \tag{15}$$

Remembering that $\varrho_2{}^2 = i \frac{\sigma - \lambda}{\nu}$, we obtain

$$\frac{1 - \frac{\tanh \varrho_2 h}{\varrho_2 h}}{\sigma - \lambda} = \frac{1}{3} \frac{ih^2}{\nu} \left(1 + \frac{2}{5} \frac{\sigma - \lambda}{\nu} h^2 \ldots\ldots\right), \tag{16}$$

and for $\sigma = \lambda$ the limit is $\frac{ih^2}{3\nu}$.

Introducing this value we get:

$$\lim \frac{2\sigma(1-\delta) + 2i\sigma\varepsilon}{\sigma^2 - \lambda^2} = \frac{1 - a_1 + ib_1}{2\sigma} + \frac{ih^2}{3\nu}, \tag{17}$$

or separating the real and the imaginary part we find:

$$\lim \frac{1-\delta}{\sigma - \lambda} = \frac{1 - a_1}{2\sigma}, \qquad \lim \frac{\varepsilon}{\sigma - \lambda} = \frac{b_1 + \frac{2\sigma h^2}{3\nu}}{2\sigma} = \frac{b_1 + b_2}{2\sigma}. \tag{18}$$

The equations for p and ϑ in this case become:

$$(1-p^2)\vartheta^2 = \theta^2 + \frac{4(1-a_1)}{(1-a_1)^2 + (b_1+b_2)^2}, \quad 2p\vartheta^2 = \frac{4(b_1+b_2)}{(1-a_1)^2 + (b_1+b_2)^2}. \tag{19}$$

To obtain the expressions for U and V, we must find the limit of

$$\frac{1 - \frac{\cosh \varrho_2 (h-z)}{\cosh \varrho_2 h}}{\sigma - \lambda}. \tag{20}$$

This limit may easily be found in the same manner as above and the result is

$$\frac{i}{2\sigma} \frac{2\sigma h^2}{\nu} \left(\frac{z}{h} - \frac{1}{2} \frac{z^2}{h^2}\right). \tag{21}$$

The expressions for φ_1 and ψ_1 are unaltered.

To get the solution in this case, we may put

$$A = \frac{\vartheta + \theta}{2}, \quad B = \frac{\vartheta - \theta}{2}, \quad A_1 = \frac{p\vartheta}{2} = B_1. \tag{22}$$

Further we have $1 - \varphi_2 = 0$ and for ψ_2 we get the expression

$$\frac{2\sigma h^2}{\nu} \left(\frac{z}{h} - \frac{1}{2} \frac{z^2}{h^2}\right). \tag{23}$$

We then obtain:

$$U = \sqrt{\frac{g}{h}}\,\frac{C}{2}\left[M\cos\sigma\left(t-\frac{x}{\varkappa}\right)+N\sin\sigma\left(t-\frac{x}{\varkappa}\right)\right]e^{-\mu x-my},$$
$$V = \sqrt{\frac{g}{h}}\,\frac{C}{2}\left[P\cos\sigma\left(t-\frac{x}{\varkappa}\right)+Q\sin\sigma\left(t-\frac{x}{\varkappa}\right)\right]e^{-\mu x-my}, \tag{24}$$

where

$$M = A(1-\varphi_1)+A_1\psi_1+B_1\psi_2,\quad N = -A\psi_1+A_1(1-\varphi_1)-B\psi_2,$$
$$P = A\psi_1-A_1(1-\varphi_1)-B\psi_2,\quad Q = A(1-\varphi_1)+A_1\psi_1-B_1\psi_2. \tag{25}$$

To obtain the solution in the case $\sigma<\lambda$ we must have recourse to the original equations. We have still

$$\varrho_1 = \sqrt{\frac{\sigma+\lambda}{2\nu}}(1+i) = \beta_1(1+i),$$

but for ϱ_2 we must put

$$\varrho_2 = \sqrt{\frac{\lambda-\sigma}{2\nu}}(1-i) = \beta_2'(1-i),$$

and

$$\frac{\tanh\varrho_2 h}{\varrho_2 h} = \frac{\sinh 2\beta_2' h+\sin 2\beta_2' h}{2\beta_2' h(\cosh 2\beta_2' h+\cos 2\beta_2' h)}+i\frac{\sinh 2\beta_2' h-\sin 2\beta_2' h}{2\beta_2' h(\cosh 2\beta_2' h+\cos 2\beta_2' h)}$$
$$= a_2'+ib_2'. \tag{26}$$

The calculations are unaltered and we only have to replace a_2 and b_2 by a_2' and $-b_2'$ in the previous solution. The sign of ε is reversed so that the sign of p is unaltered as it ought to be.

The solutions for $\sigma<\lambda$ and $\sigma>\lambda$ is therefore a continuous function of $s=\frac{\lambda}{\sigma}$.

We thus see that the solution will apply to all parts of the Arctic Ocean if the boundaries make a slope of the wave-crests possible.

8. Discussion of the Solution.

To get a clear representation of the currents we put:

$$U_1 = \sqrt{\frac{g}{h}}\,\frac{C}{2}(A-iA_1)(1-\varphi_1+i\psi_1),$$
$$V_1 = -iU_1,$$
$$U_2 = \sqrt{\frac{g}{h}}\,\frac{C}{2}(B-iB_1)(1-\varphi_2+i\psi_2),$$
$$V_2 = iU_2.$$

Further we put

$$A-iA_1 = \mathcal{A}e^{-i\gamma_1},\quad B-iB_1 = \mathcal{B}e^{-i\gamma_2},$$
$$1-\varphi_1+i\psi_1 = \mathcal{E}_1 e^{i\omega_1},\quad 1-\varphi_2+i\psi_2 = \mathcal{E}_2 e^{i\omega_2},$$

and get

$$
\begin{aligned}
u_1 &= \sqrt{\frac{g}{h}}\,\frac{C}{2}\,\mathcal{A}\mathcal{E}_1 \cos\left(\sigma t - \frac{\sigma x}{\varkappa} + \omega_1 - \gamma_1\right) e^{-\mu x - my},\\
v_1 &= \sqrt{\frac{g}{h}}\,\frac{C}{2}\,\mathcal{A}\mathcal{E}_1 \sin\left(\sigma t - \frac{\sigma x}{\varkappa} + \omega_1 - \gamma_1\right) e^{-\mu x - my},\\
u_2 &= \sqrt{\frac{g}{h}}\,\frac{C}{2}\,\mathcal{B}\mathcal{E}_2 \cos\left(\sigma t - \frac{\sigma x}{\varkappa} + \omega_2 - \gamma_2\right) e^{-\mu x - my},\\
v_2 &= -\sqrt{\frac{g}{h}}\,\frac{C}{2}\,\mathcal{B}\mathcal{E}_2 \sin\left(\sigma t - \frac{\sigma x}{\varkappa} + \omega_2 - \gamma_2\right) e^{-\mu x - my}.
\end{aligned}
\tag{1}
$$

Here we have

$$
\tan\gamma_1 = \frac{A_1}{A} = \frac{p\,\vartheta}{\vartheta + s\,\theta}, \quad \tan\gamma_2 = \frac{B_1}{B} = \frac{p\,\vartheta}{\vartheta - s\,\theta}. \tag{2}
$$

From these we obtain:

$$
\tan(\gamma_1 + \gamma_2) = \frac{p\,\vartheta\left(\dfrac{1}{\vartheta + s\,\theta} + \dfrac{1}{\vartheta - s\,\theta}\right)}{1 - \dfrac{p^2\,\vartheta^2}{\vartheta^2 - s^2\,\theta^2}} = \frac{2\,p\,\vartheta^2}{\vartheta^2(1-p^2) - s^2\,\theta^2} = \frac{\varepsilon}{1-\delta}. \tag{3}
$$

We see that u_1 and v_1 are the components of a vector which rotates in the positive direction, and u_2 and v_2 represent a vector which rotates in the negative direction. As $\mathcal{A}$ has the denominator $1 + s$ and $\mathcal{B}$ the denominator $1 - s$, we generally have $\mathcal{A} < \mathcal{B}$ and the resulting motion will be elliptic. The vector representing the current will rotate in the negative direction (clockwise).

The major semi-axis of the ellipse is

$$
\sqrt{\frac{g}{h}}\,\frac{C}{2}\,(\mathcal{A}\mathcal{E}_1 + \mathcal{B}\mathcal{E}_2),
$$

and the minor semi-axis is

$$
\sqrt{\frac{g}{h}}\,\frac{C}{2}\,(\mathcal{A}\mathcal{E}_1 - \mathcal{B}\mathcal{E}_2).
$$

The last term is negative when the rotation is negative.

As $\mathcal{E}$ and ω are functions of the depth, not only the dimensions, but also the orientation of the ellipse and the phase will vary with the depth.

If we write

$$
\Phi = \frac{\cosh\varrho\,(h-z)}{\cosh\varrho\,h} = e^{-\rho z} + \frac{e^{-\rho h}\sinh\varrho\,z}{\cosh\varrho\,h} \tag{4}
$$

we see that when ϱh is large Φ may be put approximately $e^{-\rho z}$, or

$$
\varphi = e^{-\beta z}\cos\beta z, \quad \psi = e^{-\beta z}\sin\beta z. \tag{5}
$$

We then get:

$$\mathcal{E} = \sqrt{1 - 2\,e^{-\beta z}\cos\beta z + e^{-2\beta z}}\,, \tag{6}$$

and

$$\mathcal{E}\frac{d\mathcal{E}}{dz} = \beta e^{-\beta z}\left(\cos\beta z + \sin\beta z - e^{-\beta z}\right). \tag{7}$$

$\frac{d\mathcal{E}}{dz}$ is positive for $\beta z < 2.28$, but from this value $\left|\frac{d\mathcal{E}}{dz}\right|$ will be so small that $\mathcal{E}$ is practically constant. Furthermore we see that $\frac{d\mathcal{E}}{dz}$ will increase with β and we, therefore, conclude that $\mathcal{E}_1$ increases faster than $\mathcal{E}_2$ from the bottom and upwards.

If we consider the ratio

$$\alpha = \frac{\mathcal{A}\mathcal{E}_1 - \mathcal{B}\mathcal{E}_2}{\mathcal{A}\mathcal{E}_1 + \mathcal{B}\mathcal{E}_2} \tag{8}$$

we, therefore, see that $|\alpha|$ increases from the bottom and upwards. The limiting value of α at the bottom may easily be found. The limit of

$$\frac{\mathcal{E}_1}{\mathcal{E}_2} \quad \text{is} \quad \frac{\beta_1^2}{\beta_2^2}\sqrt{\frac{a_1^2 + b_1^2}{a_2^2 + b_2^2}},$$

and the value of $\frac{\mathcal{A}}{\mathcal{B}}$ may be written $\frac{(1-s)\sin\gamma_2}{(1+s)\sin\gamma_1}$,

we then find

$$\alpha_0 = \frac{\sqrt{a_1^2 + b_1^2}\,\sin\gamma_2 - \sqrt{a_2^2 + b_2^2}\,\sin\gamma_1}{\sqrt{a_1^2 + b_1^2}\,\sin\gamma_2 + \sqrt{a_2^2 + b_2^2}\,\sin\gamma_1}. \tag{9}$$

For the total velocity we obtain:

$$\mathrm{V}^2 = (u_1 + u_2)^2 + (v_1 + v_2)^2 \tag{10}$$

$$= \frac{g}{h}\frac{\mathrm{C}^2}{4}\left[\mathcal{A}^2\mathcal{E}_1^{\,2} + \mathcal{B}^2\mathcal{E}_2^{\,2} + 2\mathcal{A}\mathcal{B}\mathcal{E}_1\mathcal{E}_2\cos\left(2\sigma t - \frac{2\sigma x}{\varkappa} + \omega_1 + \omega_2 - \gamma_1 - \gamma_2\right)\right]e^{-2\varphi x - 2my}.$$

Without loss of generality we may put $x = 0$ and $y = 0$, and the time of maximum current is then determined by

$$2\sigma t + \omega_1 + \omega_2 - \gamma_1 - \gamma_2 = 0.$$

We denote this special value of σt by τ, and we then have

$$\tau = -\frac{\omega_1 + \omega_2 - \gamma_1 - \gamma_2}{2}. \tag{11}$$

As $\tan\omega_1 = \frac{\psi_1}{1-\varphi_1}$, $\tan\omega_2 = \frac{\psi_2}{1-\varphi_2}$ and $\tan(\gamma_1 + \gamma_2) = \frac{\varepsilon}{1-\delta}$

we find the remarkable result that τ is independent of θ.

The orientation of the ellipse is determined by the angle χ which the major axis forms with the x-axis.

χ is determined by

$$\tan \chi = \left(\frac{v_1 + v_2}{u_1 + u_2}\right)_{\sigma t = \tau} = \tan \frac{\omega_1 - \gamma_1 - \omega_2 + \gamma_2}{2},$$

$$\text{or} \qquad \chi = -\frac{(\omega_2 - \gamma_2) - (\omega_1 - \gamma_1)}{2}. \tag{12}$$

The values of ω_1 and ω_2 at the bottom are determined by

$$\lim (\tan \omega_1) = \lim \left(\frac{\psi_1}{1 - \varphi_1}\right) = \frac{a_1}{b_1}, \qquad \text{and} \quad \lim (\tan \omega_2) = \frac{a_2}{b_2}.$$

For moderate depths ω_1 will be about 45° at the bottom, but ω_2 is greater. In the case $s = 1$, we have $\omega_2 = \text{const.} = 90°$ and for $s > 1$, ω_2 will be greater than 90°.

We now wish to find the variations of ω_1 and ω_2 with the depth. We have

$$\frac{d\omega}{dz} = \frac{\psi' - (\varphi\psi' - \varphi'\psi)}{(1 - \varphi)^2 + \psi^2}. \tag{13}$$

At the bottom this expression is indeterminate but has the limit

$$\frac{\beta^2 \varphi'(0)}{\varphi'(0)^2 + \psi'(0)^2} = -\frac{b}{2h(a^2 + b^2)}. \tag{14}$$

At the bottom we therefore have $\frac{d\omega}{dz} < 0$, so that ω decreases from the bottom and upwards.

If the depth is so great that we may put

$$\varphi = e^{-\beta z} \cos \beta z \qquad \psi = e^{-\beta z} \sin \beta z, \tag{15}$$

we have

$$\frac{d\omega}{dz} = -\beta \frac{e^{-\beta z}(\sin \beta z + e^{-\beta z} - \cos \beta z)}{1 - 2e^{-\beta z} \cos \beta z + e^{-2\beta z}}. \tag{16}$$

From this formula we find that $\frac{d\omega}{dz} < 0$ for $\beta z < 2.42$ so that we may conclude that ω decreases from the bottom and upwards, and also that ω_1 decreases faster than ω_2.

As we have

$$\tau = -\frac{\omega_1 + \omega_2 - \gamma_1 - \gamma_2}{2}, \tag{17}$$

we find $2\frac{d\tau}{dz} = -\frac{d\omega_1}{dz} - \frac{d\omega_2}{dz}$, or as $\frac{d\omega}{dz} < 0$ we conclude $\frac{d\tau}{dz} > 0$.

τ is generally negative at the bottom and as $\frac{d\tau}{dz} > 0$, the numerical value of τ will decrease. We thus see that maximum velocity occurs before high water and that the phase-difference is greatest at the bottom.

The orientation of the ellipse was determined by the angle χ, and for this we have

$$2\frac{d\chi}{dz} = \frac{d\omega_1}{dz} - \frac{d\omega_2}{dz}. \tag{18}$$

We see that $\frac{d\chi}{dz}$ is negative and as χ is small at the bottom it will soon become negative. The ellipse is, therefore, deflected to the right. This deflection increases from the bottom and upwards, and if the depth is great it may attain a maximum when

$$\frac{d\omega_1}{dz} = \frac{d\omega_2}{dz}.$$

This is certainly the case if h is so great that $\frac{d\omega_1}{dz}$ changes its sign, that means if $\beta h > 3{,}94$.

From Ekmans theory of gradient currents we know that the influence of the eddy viscossity is confined to a layer near the bottom. Ekman has introduced a quantity

$$D = \pi \sqrt{\frac{2\nu}{\lambda}} \tag{19}$$

which he has termed "depth of frictional influence". We know that the whole mass of water above $z = D$ will move practically as if no resistance acted. In the case of wave motion two quantities enter which play the analogous role as D, namely

$$D_1 = \pi \sqrt{\frac{2\nu}{\lambda + \sigma}}, \quad D_2 = \pi \sqrt{\frac{2\nu}{\sigma - \lambda}}. \tag{20}$$

These quantities indicate the value of z where the influence of eddy viscossity practically vanishes on the vectors (u_1, v_1) and (u_2, v_2) respectively. While $D_1 < D$ we have $D_2 > D$, so that on the whole the influence of eddy viscossity reaches to greater distances from the bottom in the case of tidal currents.

The velocity of propagation $\varkappa$ and the exponent of damping p are greatly influenced by the rotation and the friction. But as we have retained the indeterminate form of m, $\varkappa$ and p will depend not only on s and ν, but also on θ. As a general rule $\varkappa$ increases with s and decreases with θ and ν. p will increase with s and ν, and decrease with θ. That means that if the ellipses are narrow, $\varkappa$ and p are comparatively small, but they increase when the ellipses get more circular in form. For $\theta = 0$ the variation of $\varkappa$ and p has been discussed by Sverdrup.[1]) It is to be noted that the velocity of propagation by Sverdrup is denoted by c, which here always stands for the normal velocity $\sqrt{gh}$.

9. Numerical Examples.

In order to illustrate the motion a numerical example will be desireable. Suppose that the latitude is 72° N and that the depth is 50 m. We then have

$$\sigma = 1.4052 \cdot 10^{-4} \qquad \lambda = 1.387 \cdot 10^{-4} \qquad s = 0.987.$$

Further we assume that the coefficient of eddy viscossity is $\nu = 182$. If we take θ^2 to be 0.19, we obtain

$$a_1 = b_1 = 0.1153, \qquad a_2 = 0.992, \qquad b_2 = 0.081,$$
$$p = 0.438, \qquad \vartheta = 0.58.$$
$$A = 0.508, \qquad B = 11.41, \qquad A_1 = 0.128, \qquad B_1 = 19.491.$$

[1]) L. c.

The solution is

$$u = \sqrt{\frac{g}{h}}\,\frac{C}{2}\left[M\cos\sigma\left(t-\frac{x}{\varkappa}\right)+N\sin\sigma\left(t-\frac{x}{\varkappa}\right)\right]e^{-\mu x-my},$$

$$v = \sqrt{\frac{g}{h}}\,\frac{C}{2}\left[P\cos\sigma\left(t-\frac{x}{\varkappa}\right)+Q\sin\sigma\left(t-\frac{x}{\varkappa}\right)\right]e^{-\mu x-my}.$$

To find the dimensions and the orientation of the ellipse for $x = 0$, $y = 0$, we also have the formulæ

$$\begin{aligned} 2V &= \sqrt{\frac{g}{h}}\,\frac{C}{2}\left(\sqrt{(M+Q)^2+(N-P)^2}+\sqrt{(M-Q)^2+(N+P)^2}\right),\\ 2\alpha V &= \sqrt{\frac{g}{h}}\,\frac{C}{2}\left(\sqrt{(M+Q)^2+(N-P)^2}-\sqrt{(M-Q)^2+(N+P)^2}\right), \end{aligned} \tag{1}$$

$$\tan 2\chi = \frac{2(MP+NQ)}{(M+Q)(M-Q)+(N+P)(N-P)},$$

$$\tan 2\tau = \frac{2(MN+PQ)}{(M+Q)(M-Q)-(N+P)(N-P)},$$

where V is the major semi-axis of the ellipse and αV the minor semi-axis.

Using these formulae the following table has been calculated.

d	0	20	40	45
φ_1	− 0.0098	− 0.0632	0.2717	0.5882
ψ_1	− 0.0244	0.038	0.320	0.2723
φ_2	0.9876	0.9899	0.9961	0.998
ψ_2	0.1213	0.1019	0.0437	0.023
M	3.015	2.6461	1.449	0.715
N	− 1.001	− 0.8196	− 0.549	− 0.308
P	− 1.283	− 1.0832	− 0.415	− 0.138
Q	− 1.995	− 1.5561	− 0.627	− 0.227
χ	− 20°	− 18°.4	− 7°.5	− 3°
τ	− 5°	− 6°.3	− 17°	− 22°.6
t (min)	10	13	35	46
V	3 28	2.87	1.56	0.78
αV	− 2.22	− 1.76	− 0.73	− 0.26
α	− 0.68	− 0.61	− 0.47	− 0.34

As another example we may take $\lambda = 1.4412 \cdot 10^{-4}$ which corresponds to $\varphi =$ 81° 11′ N Lat.

We then obtain:

$$\beta_1 = 0.8893 \cdot 10^{-3} \qquad \beta'_2 = 10^{-4}.$$

With $h = 5 \cdot 10^3$ cm. we further find

$$a_1 = b_1 = \frac{1}{8.893}, \qquad a_2 = 0.968, \qquad b_2 = -0.1602,$$

$$\varepsilon = 0.9789, \qquad \delta = -0.1637.$$

Assuming $\vartheta^2 = 0.19$, we find $p = 0.611$, $\vartheta = 0.505$, $A = 0.470$, $A_1 = 0.152$, $B = -2.258$, $B_1 = -12.051$. The results for the different depths are grouped in the table below.

d	0	20	40	48
φ_1	— 0.062	— 0.078	0.2589	0.8239
ψ_1	— 0.0226	0.032	0.3192	0.1481
φ_2	0.95	0.96	0.984	0.9968
ψ_2	— 0.2398	— 0.2027	— 0.087	— 0.019
M	3.247	2.832	1.397	0.327
N	— 0.980	— 0.789	— 0.426	— 0.135
P	— 1.308	— 1.087	— 0.352	— 0.039
Q	— 2.307	— 1.808	— 0.607	— 0.115
χ	— 20°.8	— 19°.1	— 7°.9	+ 2°
τ		— 3°.5	— 13°.2	— 22°.9
t (min)		7	27	47
V	3.498	3.035	1.472	0.354
α V	— 2.506	— 1.969	— 0.679	— 0.121
α	— 0.72	— 0.65	— 0.53	— 0.34

A comparison of the two tables shows that the configuration of the motion is on the whole the same. The current-vector rotates clockwise, and the greatest velocities occur at the surface. The minimum velocity decreases faster than the maximum velocity, so that the ratio α is greatest at the surface and decreases as we go downwards.

The maximum velocity occurs before high water, but the time-difference decreases from the bottom to the surface. At the surface high water and maximum velocity occur almost simultaneously. The major axis of the ellipse forms an angle with the x-axis which was denoted by χ. This angle is at the surface about 20° and decreases towards the bottom. But if we compare the velocities in the two cases we see that there is some difference, the increase of the velocity being slower in the first case.

If we consider the quantities

$$D_1 = \pi \sqrt{\frac{2\nu}{\sigma + \lambda}}, \quad D_2 = \pi \sqrt{\frac{2\nu}{\sigma - \lambda}} \quad \text{or} \quad D_2' = \pi \sqrt{\frac{2\nu}{\lambda - \sigma}}$$

we have in the first case $D_1 = 36.2$ m. $D_2 = 444.3$ m.
and in the second case $D_1 = 35.3$ m. $D_2' = 314.2$ m.

These quantities indicate the distances from the bottom to which the influence of the viscossity is perceptible, D_1 on φ_1 and ψ_1, and D_2 on φ_2 and ψ_2.

We see that D_2 is greatest in the first case and this explains why the increase of the velocity near the surface is slower. The influence of the eddy viscossity must reach its maximum in the case $s = \frac{\lambda}{\sigma} = 1$ when $D_2 = \infty$.

10. Ice-covered Sea.

If the ice does not take part in the movement we may put $fh = \infty$ and then we find:

$$\varphi_1 - i\psi_1 = \frac{\sinh \varrho_1 (h - z) + \sinh \varrho_1 z}{\sinh \varrho_1 h} = \frac{\cosh \varrho_1 \left(\frac{h}{2} - z\right)}{\cosh \varrho_1 \frac{h}{2}}. \tag{1}$$

We obtain:

$$\begin{aligned} \varphi_1 &= \frac{\cosh \beta_1 (h - z) \cos \beta_1 z + \cosh \beta_1 z \cos \beta_1 (h - z)}{\cosh \beta_1 h + \cos \beta_1 h}, \\ \psi_1 &= \frac{\sinh \beta_1 (h - z) \sin \beta_1 z + \sinh \beta_1 z \sin \beta_1 (h - z)}{\cosh \beta_1 h + \cos \beta_1 h}, \end{aligned} \tag{2}$$

and the analogous expressions for φ_2 and ψ_2.

For a and b the expressions will be:

$$a_1 = \frac{1}{\beta_1 h} \frac{\sinh \beta_1 h + \sin \beta_1 h}{\cosh \beta_1 h + \cos \beta_1 h}, \quad b_1 = \frac{1}{\beta_1 h} \frac{\sinh \beta_1 h - \sin \beta_1 h}{\cosh \beta_1 h + \cos \beta_1 h}, \tag{3}$$

and the analogous formulæ for a_2 and b_2.

The solution of the equation of continuity is as in the preceding case only that $2\beta h$ here is replaced by βh. The influence of the resistance is therefore greatly increased.

In the case $s = 1$, we have $1 - a_1 = 0$ and $b_2 = \frac{1}{6} \frac{\sigma h^2}{\nu}$, and ψ_2 is to be replaced by

$$\frac{\sigma h^2}{\nu} \left(\frac{z}{h} - \frac{z^2}{h^2}\right). \tag{4}$$

To see how the effect of the resistance is increased, we may compare a numerical solution in this case with the preceding one which was derived at under the assumption of a free surface.

In the preceding case we had $\varphi = 72°$ N and $\nu = 182$, and we found $p = 0.438$.

In this case we find $p = 0.667$.

To see the effect more clearly we may calculate the fraction to which the original amplitude is reduced when the wave has advanced a distance of 555 km., corresponding to the distance across the Siberian continental shelf. In the case of a free surface we find that a tidal range of 18 cm. would be reduced to 7.4 cm. In the case of a resting ice-covered surface the range would be reduced to 2.1 cm. This indicates that there may be a great seasonal change in the tides along the Siberian coasts, as already pointed out by Sverdrup[1])

The general case when $fh = 0$ may be brought in relation to the current measurements of Professor Sverdrup at his Stations 7, 8 and 9, and we shall, therefore, develope the equations so far that they may be used for numerical computations.

[1]) L. c. pag. 75.

If we split the hyperbolic functions in their real and imaginary parts, the expressions will be imensely long and untractable, and we therefore proceed in the following manner:

In the case $fh = 0$ we had

$$a - ib = \frac{\tanh \varrho h}{\varrho h}, \quad \text{say A}.$$

And if $fh = \infty$ we had

$$a - ib = \frac{2 \tanh \frac{\varrho h}{2}}{\varrho h} \quad \text{say B}.$$

In the general case we may therefore put

$$a - ib = \frac{\mathrm{A}(1 + fh\mathrm{B})}{1 + fh\mathrm{A}}, \tag{5}$$

or if we put $\mathrm{A} = a' - ib', \quad \mathrm{B} = a'' - ib'',$

we get:

$$\begin{aligned} a &= a'' + \frac{(1 + fha')(a' - a'') + fhb'(b' - b'')}{(1 + fha')^2 + (fhb')^2}, \\ b &= b'' + \frac{(1 + fha')(b' - b'') - fhb'(a' - a'')}{(1 + fha')^2 + (fhb')^2}. \end{aligned} \tag{6}$$

In the same way we may put:

$$\begin{aligned} \varphi &= \varphi'' + \frac{(1 + fha')(\varphi' - \varphi'') + fhb'(\psi' - \psi'')}{(1 + fha')^2 + (fhb')^2}, \\ \psi &= \psi'' + \frac{(1 + fha')(\psi_{,}' - \psi_{,}'') - fhb'(\varphi_{,} - \varphi'')}{(1 + fha')^2 + (fhb')^2}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \varphi_1' &= \frac{\cosh \beta_1(2h - z)\cos \beta_1 z + \cosh \beta_1 z \cos \beta_1(2h - z)}{\cosh 2\beta_1 h + \cos 2\beta_1 h}, \\ \psi_1' &= \frac{\sinh \beta_1(2h - z)\sin \beta_1 z + \sinh \beta_1 z \sin \beta_1(2h - z)}{\cosh 2\beta_1 h + \cos 2\beta_1 h}. \end{aligned} \tag{8}$$

To get φ_1'' and ψ_1'', we replace $2h$ by h. φ_2', ψ_2', φ_2'' and ψ_2'' have analogous expressions only that β_1 is replaced by β_2.

If we have calculated these quantities we may build up the solution as before. In the case $s = 1$ these equations admit great simplifications. We then have $\varphi_2 = 1$ and for ψ_2 we find the limit:

$$\psi_2 = \frac{\sigma h^2}{\nu}\left(\frac{z}{h}\frac{2 + fh}{1 + fh} - \frac{z^2}{h^2}\right). \tag{9}$$

Similarly we have $1 - a_2 = 0$ and $b_2 = \frac{1}{6}\frac{\sigma h^2}{\nu}\frac{4 + fh}{1 + fh}$. (10)

as it is easy to show.

We remember that

$$\lim_{s=1}\left(\frac{b_2'}{1-s}\right)=\frac{h^2}{3\nu}, \qquad \text{and} \qquad \lim\left(\frac{b_2''}{1-s}\right)=\frac{h^2}{12\nu},$$

and that we, therefore in the case $s=1$ put

$$b_2'=\frac{2}{3}\frac{\sigma h^2}{\nu}, \qquad b_2''=\frac{1}{6}\frac{\sigma h^2}{\nu}.$$

We also remember that $\lim a_2'=1$ and $\lim b_2'=0$, and we get

$$\lim_{s=1}\left(\frac{b_2}{1-s}\right)=\lim\frac{b_2''}{1-s}+\lim\frac{(1+fha_2')\dfrac{b_2'-b_2''}{1-s}-fh\dfrac{b_2'}{1-s}(a_2'-a_2'')}{(1+fha_2')^2+(fhb_2')^2} \tag{11}$$

$$=\frac{1}{12}\frac{h^2}{\nu}+\frac{1+fh\left(\dfrac{h^2}{3\nu}-\dfrac{h^2}{12\nu}\right)}{(1+fh)^2}=\frac{1}{12}\frac{h^2}{\nu}\frac{4+fh}{1+fh},$$

and as B_1 and B_2 have been divided by 2σ, the above expression has to be multiplied by 2σ to get the value of b_2 to be used when $s=1$.

The limit for ψ is easily found in the same manner.

11. Slipping Motion along the Bottom.

The assumption of no slipping along the bottom seems to answer rather well to the observed currents in most cases but it may nevertheless be of interest to see if the currents are much modified when we allow slipping motion along the bottom. This point has been investigated by H. U. Sverdrup.[1]) As boundary-conditions he has chosen:

$$\left.\begin{aligned}\frac{\partial u}{\partial t}-\lambda v+ku&=-g\frac{\partial\zeta}{\partial y}\\ \frac{\partial v}{\partial t}+\lambda u+kv&=0\end{aligned}\right\}\text{when } z=0. \tag{1}$$

Taking account of the equations of motion, these conditions may be written in the more simple form:

$$\left.\begin{aligned}\nu\frac{\partial^2 u}{\partial z^2}+ku&=0\\ \nu\frac{\partial^2 v}{\partial z^2}+kv&=0\end{aligned}\right\}\text{when } z=0. \tag{2}$$

He is then led to the conclusion that the configuration of the motion is even more modified close to the bottom than in the case assuming no slip along the bottom. He finds that the ratio between the axes of the ellipse is more reduced and that the ellipse is turned more to the right. Further he is led to an equation of condition which binds together $\frac{k}{\sigma}$ and the maximal value of p.

[1]) L. c. p. 44.

But I cannot agree with his boundary conditions and I have therefore replaced them by the following:

$$\frac{\partial u}{\partial z} = fu\,,\ \frac{\partial v}{\partial z} = fv\,,\ z = 0$$

The value $f = \infty$ then corresponds to no slip at the boundary.
The boundary-conditions give:

$$\frac{dU_1}{dz} - fU_1 = 0\,, \qquad \frac{dU_2}{dz} - fU_2 = 0\,, \tag{3}$$

and at the surface we take.

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0\,,\ z = h\,.$$

We then easily find:

$$U_1 = A_0\,(1 - \Phi_1),\ U_2 = B_0\,(1 - \Phi_2)\,, \tag{4}$$

where

$$\Phi = \frac{f \cosh \varrho\,(h - z)}{f \cosh \varrho\, h + \varrho \sinh \varrho\, h}\,. \tag{5}$$

For $A = \dfrac{1}{h}\displaystyle\int_0^h \Phi\, dz$, we get:

$$A = \frac{1}{\varrho h}\,\frac{f \sinh \varrho\, h}{f \cosh \varrho\, h + \varrho \cosh \varrho\, h}\,. \tag{6}$$

To split Φ and A in their real and imaginary parts we put as in the case $f = \infty$,

$$\varphi - i\psi = \frac{\cosh \varrho\,(h - z)}{\cosh \varrho h}\,, \qquad a - ib = \frac{1}{\varrho h} \tanh \varrho h\,, \tag{7, 8}$$

and then we get:

$$\Phi = \frac{\varphi - i\psi}{1 + \dfrac{\varrho^2 h^2}{fh}\,(a - ib)}\,, \qquad A = \frac{a - ib}{1 + \dfrac{\varrho^2 h^2}{fh}\,(a - ib)}\,. \tag{9}$$

Or if we put:

$$\frac{\varrho^2 h^2}{fh} = \frac{2i\beta^2 h^2}{fh} = in\,,$$

$$\begin{aligned} \Phi &= \frac{\varphi - i\psi}{1 + nb + ina} = \frac{\varphi\,(1 + nb) - \psi na}{(1 + nb)^2 + n^2 a^2} - i\,\frac{\psi\,(1 + nb) + \varphi na}{(1 + nb)^2 + n^2 a^2}\,, \\ A &= \frac{a - ib}{1 + nb + ina} = \frac{a}{(1 + nb)^2 + n^2 a^2} - i\,\frac{b\,(1 + nb) + na^2}{(1 + nb)^2 + n^2 a^2}\,. \end{aligned} \tag{10}$$

The calculations are otherwise unaltered. For comparison we may compute a numerical example.

We assume $s = \dfrac{\lambda}{\sigma} = 0.96$, $h = 50$ m. $= 5 \cdot 10^3$ cm.

$$\beta_1 = 0.7 \cdot 10^{-3}\,, \quad \beta_2 = 1 \cdot 10^{-4}\,, \quad fh = 3.5 \text{ and } \theta = 0\,.$$

We then get: $a_1 = 0.14283$, $a_2 = 0.96799$,
$b_1 = 0.14249$, $b_2 = 0.16018$,

$$n_1 = 7, \qquad n_2 = \frac{1}{7},$$

$$A_1 = 0.02857 - i\,0.08572, \quad A_2 = 0.90856 - i\,0.27943.$$

With these values we find:

$$p = 0.7415, \quad \vartheta = 0.5503, \quad \varkappa = 40.27.$$

Further we may compute the following table giving relative values of the velocity and the configuration of the motion at the bottom and at the surface.

The values referring to slipping motion at the bottom has been provided with an index *f*.

z	V_f	V	$-\alpha V_f$	$-\alpha V$	$-\alpha_f$	$-\alpha$	χ_f	χ	τ_f	τ
0	1.91	0	1.58	0	0.8262	0.6589	$-27°.3$	$-17°.8$	$-11°.4$	$-27°.4$
h	5.09	4.57	4.51	3.82	0.8855	0.8354	$-36°.3$	$-39°.7$	$-1°.0$	$-2°.0$

We see that the effect of the viscossity is on the whole reduced. The configuration is approximately as if the depth had been greater, and the lowest part of the current curve had been cut away. The deflection of the current vectors near the bottom is greater when there is slipping motion than in the other case, but this is easily understood by the remark above, as χ under ordinary conditions increases from the bottom and upwards.

12. Eddy Viscossity a Function of Depth.

The assumption of constant eddy viscossity is only an approximation. In reality the viscossity must depend upon the hydrographic conditions, especially on the stability of the water-layers. If we denote the density by ϱ, the viscossity must be some function of $\frac{d\varrho}{dz}$, and such that great values of $\frac{d\varrho}{dz}$ correspond to small values of ν and vice versa. The function $\nu = f\left(\frac{d\varrho}{dz}\right)$ is yet unknown, but we may still to some degree find out how the preceding conclusions will be modified if the eddy viscossity varies with the depth.

The equations of motion take the form:

$$\begin{aligned} \frac{\partial v}{\partial t} - \lambda v &= -g\frac{\partial \zeta}{\partial x} + \frac{\partial}{\partial z}\left(\nu \frac{\partial u}{\partial z}\right), \\ \frac{\partial v}{\partial t} + \lambda u &= -g\frac{\partial \zeta}{\partial y} + \frac{\partial}{\partial z}\left(\nu \frac{\partial v}{\partial z}\right). \end{aligned} \tag{1}$$

The equation of continuity is of course unaltered.

We assume as before

$$u = U(z)e^{i\sigma\left(t - \frac{x}{\varkappa}\right) - \mu x - my} \text{ etc.} \tag{2}$$

and then we get:

$$\begin{aligned} i\sigma U - \lambda V &= g\gamma C + \frac{d}{dz}\left(\nu \frac{dU}{dz}\right) \\ i\sigma V + \lambda U &= gmC + \frac{d}{dz}\left(\nu \frac{dv}{dz}\right). \end{aligned} \tag{3}$$

If we put

$$U = U_1 + U_2, \quad V = -iU_1 + iU_2, \tag{4}$$

we get the equations:

$$\begin{aligned} \frac{d}{dz}\left(\nu \frac{dU_1}{dz}\right) - i(\sigma + \lambda)U_1 &= -i\frac{m - i\gamma}{2} gC, \\ \frac{d}{dz}\left(\nu \frac{dU_2}{dz}\right) - i(\sigma - \lambda)U_2 &= i\frac{m + i\gamma}{2} gC, \qquad \nu = \nu_0 k(z). \end{aligned} \tag{5}$$

As before we put

$$\beta_1 = \sqrt{\frac{\sigma + \lambda}{2\nu_0}}, \quad \beta_2 = \sqrt{\frac{\sigma - \lambda}{2\nu_0}}, \quad A_1 = \frac{m - i\gamma}{\sigma + \lambda}\frac{gC}{2}, \quad A_2 = -\frac{m + i\gamma}{\sigma - \lambda}\frac{gC}{2},$$

the two equations may then be represented by:

$$\frac{d}{dz}\left(k(z)\frac{dU}{dz}\right) - 2i\beta^2 U = -2i\beta^2 A, \tag{6}$$

where we in turn replace U by U_1 and U_2 etc.

Let the integral of the homogeneous equation be

$$B_1 P_1(z) + C_1 Q(z),$$

we then obtain:

$$U_1 = A_1 + B_1 P_1(z) + C_1 Q_1(z) \tag{7}$$

and

$$U_2 = A_2 + B_2 P_2(z) + C_2 Q_2(z) \tag{8}$$

At the bottom $z = 0$ we assume $u = 0$ and $v = 0$, or $U_1 = 0$, $U_2 = 0$.

At the surface we assume $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$, i. e. $\frac{dU_1}{dz} = \frac{dU_2}{dz} = 0$.

These conditions give:

$$\begin{aligned} A + BP(0) + CQ(0) &= 0, \\ BP'(h) + CQ'(h) &= 0, \end{aligned}$$

or

$$B_1 = -A_1 \frac{Q_1'(h)}{Q_1'(h)P_1(0) - P_1'(h)Q_1(0)}, \quad C_1 = -A_1 \frac{-P_1'(h)}{Q_1'(h)P_1(0) - P_1'(h)Q_1(0)}.$$

For U_1 we then get:

$$U_1 = A_1\left(1 - \frac{Q_1'(h)P_1(z) - P_1'(h)Q_1(z)}{Q_1'(h)P_1(0) - P_1'(h)Q_1(0)}\right), \tag{9}$$

and the analogous expression for U_2.

If we introduce these values in the expressions for u and v we get:

$$U = \frac{m - i\gamma}{\sigma + \lambda}\frac{gC}{2}(1 - \Phi_1) - \frac{m + i\gamma}{\sigma - \lambda}\frac{gC}{2}(1 - \Phi_2),$$

$$V = -i\frac{m - i\gamma}{\sigma + \lambda}\frac{gC}{2}(1 - \Phi_1) - i\frac{m + i\gamma}{\sigma - \lambda}\frac{gC}{2}(1 - \Phi_2), \tag{10}$$

where

$$\Phi_1 = \frac{Q_1'(h)\,P_1(z) - P_1'(h)\,Q_1(z)}{Q_1'(h)\,P_1(0) - P_1'(h)\,Q_1(0)}, \tag{11}$$

and Φ_2 has the analogous expression only that P_1 and Q_1 is replaced by P_2 and Q_2.

The equation of continuity takes the form:

$$i\sigma C = \int_0^h (\gamma U + mV)\,dz, \tag{12}$$

or if we put:

$$A_1 = \frac{1}{h}\int_0^h \Phi_1\,dz, \quad A_2 = \frac{1}{h}\int_0^h \Phi_2\,dz. \tag{13}$$

we get:

$$-\frac{i\sigma}{c^2} = i(m^2 + \gamma^2)\left[\frac{1}{2}\frac{1 - A_1}{\sigma + \lambda} + \frac{1}{2}\frac{1 - A_2}{\sigma - \lambda}\right].$$

As

$$\int_0^h P_1\,dz = \frac{1}{2i\beta_1^2}\int_0^h \frac{d}{dz}\left(k\frac{dP}{dz}\right)dz = \frac{1}{2i\beta_1^2}\left[k(h)P'(h) - k(0)P'(0)\right],$$

we obtain

$$A_1 = \frac{i}{2\beta_1^2 h}\frac{P_1'(0)\,Q_1'(h) - Q_1'(0)\,P_1'(h)}{P_1(0)\,Q_1'(h) - Q_1(0)\,P_1'(h)}, \tag{14}$$

and the analogous expression for A_2.

The solution of the problem depends on the integration of the equation

$$\frac{d}{dz}\left(k(z)\frac{d\Phi}{dz}\right) - 2i\beta^2\Phi = 0.$$

According to the physical interpretation of the function $k(z)$, we know that it must be positive. The equation has therefore no singularity in the interval concerned, and we may find two independent integrals which are holomorphic. The general integral depends on two arbitrary constants and as we have two boundary conditions, the problem has no theoretical difficulties. But we know that the function k may approach the value zero and it will therefore also be of interest to consider the limiting case that $k(z)$ has a zero point in the interval $0 < z < h$ say for $z = h_1$. Let $k(z)$ have the form:

$$k = \left(1 - \frac{z}{h_1}\right)^n P_n\left(1 - \frac{z}{h_1}\right)$$

where P_n is a convergent series.

The sea is then divided in two different layers and the tangential traction between the two layers is nil. For $z = h_1$ we must therefore have $k\frac{d\Phi}{dz} = 0$.

Another necessary condition is that Φ shall remain finite for $z = h_1$. In general one of these conditions will involve the other. If the surface is free, Φ must be identic zero for $z > h_1$.

Two different cases will be of interest.

$$n > 2, \quad n \leqq 2.$$

If $n \leqq 2$ the equation is of Fuch's type and has one integral which remains finite for $z = h_1$, while the other is infinite. In this case we must chose the integral which is finite.

If $n > 2$ on the other hand the equation has two independent integrals which are indeterminate for $z = h_1$. If we for a moment regard z as a complex variable the value of the integral depends on the direction in which z approaches h_1. But it is still possible to find an integral which has the limit of zero when z tends to h_1 through real values.

To illustrate this we may assume

$$k(z) = \left(1 - \frac{z}{h_1}\right)^n$$

where n is an arbitrary exponent. The restriction that n shall be an integer may now be dropped. The differential-equation is now:

$$\frac{d}{dz}\left[\left(1 - \frac{z}{h_1}\right)^n \frac{d\Phi}{dz}\right] - 2i\beta^2\Phi = 0. \tag{15}$$

If $n \neq 2$, we may express the integrals by means of Bessel's functions. This can be done by introducing the new variables

$$\left(1 - \frac{z}{h_1}\right) = x^{-\frac{2}{n-2}}, \qquad \Phi = x^{\frac{n-1}{n-2}} v.$$

The equation then takes the form:

$$v'' + \frac{1}{x} v' + \left[-2i\left(\frac{2\beta h_1}{n-2}\right)^2 - \frac{\left(\frac{n-1}{n-2}\right)^2}{x^2}\right] v = 0, \tag{16}$$

which is the differential equation of Bessel's functions of order $\frac{n-1}{n-2}$.

For the direct treatment of the equation it is more convenient to introduce the new variable

$$\left(1 - \frac{z}{h_1}\right) = x^{-\frac{1}{n-2}}.$$

The equation then takes the form:

$$x\Phi'' + \frac{1}{2-n}\Phi' - 2i\left(\frac{\beta h}{2-n}\right)^2 \Phi = 0. \tag{17}$$

To integrate this equation we assume

$$\Phi = \sum a_m x^{m+s}$$

and get

$$\Phi' = \sum (m+s)\, a_m x^{m+s-1}$$

$$x\Phi'' = \sum (m+s)(m+s-1)\, a_m x^{m+s-1}$$

Introducing these expressions in the differential equation, the successive coefficients are determined by the formula:

$$(m+s)\left(m+s-\frac{1-n}{2-n}\right)a_m = 2i\left(\frac{\beta h}{2-n}\right)^2 a_{m-1} \tag{18}$$

$m=0$ gives $s=0$ and $s=\frac{1-n}{2-n}$.

Corresponding to the root $s=0$, we get

$$a_m = 2i\left(\frac{\beta h}{2-n}\right)^2 \frac{a_{m-1}}{m\left(m-\frac{1-n}{2-n}\right)} \tag{19}$$

The series will be convergent if $\frac{1-n}{2-n}$ is not an integer.

Corresponding to the other root $s=\frac{1-n}{2-n}$ we get the series:

$$x^{\frac{1-n}{2-n}} \sum b_m x^m$$

where the coefficients are found by the formula

$$b_m = 2i\left(\frac{\beta h}{2-n}\right)^2 \frac{b_{m-1}}{m\left(m+\frac{1-n}{2-n}\right)} \tag{20}$$

This series will converge if $\frac{1-n}{2-n}$ is not a negative integer. This means we exclude the following values of n

$$n = 1,\ \tfrac{3}{2},\ \tfrac{5}{3}\ \ldots\ \frac{2r+1}{r+1}\ \ldots\ .$$

One of the integrals is a power-series of x and the other is a similar power-series multiplied by the factor

$$x^{\frac{1-n}{2-n}} = \left(1-\frac{z}{h_1}\right)^{1-n}$$

As $x=\left(1-\frac{z}{h_1}\right)^{2-n}$ we see that both integrals are finite for $z=h_1$ when $n<1$. When $1<n<2$, one of the integrals is finite and the other is infinite.

If $n>2$ we have

$$x = \left(1-\frac{z}{h_1}\right)^{-(n-2)}$$

and both integrals are indeterminate for $z=h_1$. If n has one of the excluded values, one of the integrals contains a logaritm, and is therefore infinite, while the other is finite when $n<2$.

If $n=2$ we have an integral

$$\Phi = \left(1-\frac{z}{h_1}\right)^r \tag{21}$$

where r is a root of the equation

$$r(r+1) - 2i\beta^2 {h_1}^2 = 0.$$

In this case we also find that one of the integrals remains finite for $z = h_1$.

In every case we thus can find an integral which remains finite for $z = h_1$.

We have seen that if $n < 1$ both integrals remain finite for $z = h_1$. The condition which must be fulfilled is $\nu \frac{d\Phi}{dz} = 0$, and from this we see that the power-series represents the integral which is to be taken. The integral is different from zero for $z = h_1$.

When $n = 2$ the value of the integral tends to zero for $z = h_1$. The case $n > 2$ is best illustrated by taking $n = 4$. The cylinder functions then are of Poisson's type and may be expressed by elementary transcendents:

$$\Phi_1 = \varrho x e^{-\varrho x} + e^{-\varrho x}, \qquad \Phi_2 = \varrho x e^{\varrho x} - e^{\varrho x},$$

or if we again introduce z:

$$\Phi_1 = \left(\frac{\beta h_1 (1+i)}{1 - \frac{z}{h_1}} + 1\right) e^{-\frac{\beta h_1 (1+i)}{1-\frac{z}{h_1}}}, \quad \Phi_2 = \left(\frac{\beta h_1 (1+i)}{1 - \frac{z}{h_1}} - 1\right) e^{+\frac{\beta h_1 (1+i)}{1-\frac{z}{h_1}}}. \tag{20}$$

Both integrals are indeterminate for $z = h_1$.

As for Φ_1 we find that it tends to zero when z approaches h_1, resting smaller than h_1, but tends to infinity when the difference $h_1 - z$ tends to zero through negative values.

For the interval $0 < z < h_1$ we may therefore use Φ_1 and for $h_1 < z < h$ we may use Φ_2.

If the surface is free, Φ is identic zero for $z > h_1$.

Another case which may be solved by means of Bessel's functions is the case

$$\nu = \nu_0 e^{-\frac{\alpha z}{h}}$$

We then have to integrate the equation

$$\frac{d}{dz}\left(e^{-\frac{\alpha z}{h}} \frac{dU}{dz}\right) - 2i\beta^2 U = 0. \tag{22}$$

We introduce a new variable x by $x = e^{\frac{\alpha z}{2h}}$, and then we get

$$\frac{dU}{dz} = \frac{\alpha}{2h} x \frac{dU}{dx}, \qquad \frac{d}{dz}\left(e^{-\frac{\alpha z}{h}} \frac{du}{dz}\right) = \frac{\alpha^2}{4h^2} x \frac{d}{dx}\left(\frac{1}{x}\frac{du}{dx}\right).$$

The equation then becomes

$$x \frac{d}{dx}\left(\frac{1}{x}\frac{dU}{dx}\right) - \frac{8i\beta^2 h^2}{\alpha^2} U = 0. \tag{23}$$

If we now put $U = xy$ we get:

$$\frac{dU}{dx} = x\frac{dy}{dx} + y,$$

$$x \frac{d}{dx}\left(\frac{1}{x}\frac{dU}{dx}\right) = x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{1}{x}y,$$

and the equation takes the form

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(\varrho^2 - \frac{1}{x^2}\right)y = 0\,, \qquad \varrho^2 = -\frac{8i\beta^2h^2}{\alpha^2}\,. \tag{24}$$

This is the differential equation of Bessel's functions of order 1, and the integral is[1])

$$y = AH_2^1(\varrho x) + BH_1^1(\varrho x)\,,$$

and consequently:

$$U = Ax H_2^1(\varrho x) + Bx H_1^1(\varrho x)\,. \tag{24}$$

We therefore put

$$P(z) = x H_2^1\,, \qquad Q(z) = x H_1^1\,, \quad \text{etc.}$$

We then get:

$$\frac{dP}{dz} = \frac{\alpha x}{2h}\frac{d}{dx}\left(x\,H_2^1(\varrho x)\right),$$

and by the known formulae of Bessel's functions this becomes:

$$\frac{\alpha}{2h}\varrho x^2 H_2^0(\varrho x) = \beta x^2(1-i)\,H_2^0(\varrho x)\,.$$

In the same way:

$$\frac{dQ}{dz} = \beta x^2(1-i)\,H_1^0(\varrho x)\,.$$

The expression for Φ then takes the form:

$$\Phi_1 = x\,\frac{H_2^1(\varrho_1 x)H_1^0(\varrho_1 x_1) - H_1^1(\varrho_1 x)H_2^0(\varrho_1 x_1)}{H_2^1(\varrho_1)H_1^0(\varrho_1 x_1) - H_1^1(\varrho_1)H_2^0(\varrho_1 x_1)}\,, \quad x_1 = e^{\frac{\alpha}{2}}\,. \tag{25}$$

To get Φ_2 we only replace ϱ_1 by ϱ_2.

In the same manner we find the values of A_1 and A_2. We get:

$$A_1 = \frac{1+i}{2\beta h}\,\frac{H_2^0(\varrho_1)H_1^0(\varrho_1 x_1) - H_1^0(\varrho_1)H_2^0(\varrho_1 x_1)}{H_2^1(\varrho_1)H_1^0(\varrho_1 x_1) - H_1^1(\varrho_1)H_2^0(\varrho_1 x_1)}\,. \tag{26}$$

To get A_2 we have to replace β_1 and ϱ_1 by β_2 and ϱ_2.

13. Integration in the Case $s = 1$.

The equations of motion in this case take the form:

$$\begin{aligned} i\sigma U - \sigma V &= g\gamma C + \frac{d}{dz}\left(\nu\frac{dU}{dz}\right), \\ i\sigma V + \sigma U &= gmC + \frac{d}{dz}\left(\nu\frac{dV}{dz}\right). \end{aligned} \tag{1}$$

[1]) H is the Notation of N. Nielsen, See Handbuch der Theorie der Cylinderfunctionen.

As before we put:

$$U = U_1 + U_2, \quad V = -iU_1 + iU_2, \quad \text{and get:}$$

$$\begin{aligned} \frac{d}{dz}\left(\nu \frac{dU_1}{dz}\right) - 2i\sigma U_1 &= -i\frac{m-i\gamma}{2} gC = -2i\sigma A_0, \\ \frac{d}{dz}\left(\nu \frac{dU_2}{dz}\right) &= i\frac{m+i\gamma}{2} gC = -2i\sigma B_0. \end{aligned} \tag{2}$$

From the last equation we get:

$$\begin{aligned} \nu \frac{dU_2}{dz} &= -2i\sigma B_0 (z + \varkappa), \\ U_2 &= -2i\sigma B_0 \int_0^z \frac{z+\varkappa}{\nu} dz. \end{aligned} \tag{3}$$

As boundary-conditions we take $U_2 = 0, \quad z = 0, \quad \nu \frac{dU_2}{dz} = 0, \quad z = h.$

We then find

$$\nu \frac{dU_2}{dz} = 2i\sigma B_0 (z + \varkappa),$$

and if we put $z = h$, we find $\varkappa = -h$.

We put

$$2\sigma \int_0^z \frac{h-z}{\nu} dz = \psi_2. \tag{4}$$

For U_1 we get as before.

$$U_1 = \frac{m - i\gamma}{4\sigma} gC (1 - \Phi_1), \tag{5}$$

where

$$\Phi_1 = \frac{P_1(z)\, Q_1'(h) - Q_1(z)\, P_1'(h)}{P_1(0)\, Q_1'(h) - Q_1(0)\, P_1'(h)}. \tag{6}$$

The complete solution is then

$$\begin{aligned} U &= \frac{m - i\gamma}{4\sigma} gC(1 - \Phi_1) - i\frac{m + i\gamma}{4\sigma} gC\psi_2, \\ V &= -i\frac{m - i\gamma}{4\sigma} gC(1 - \Phi_1) + \frac{m + i\gamma}{4\sigma} gC\psi_2. \end{aligned} \tag{7}$$

The equation of continuity will be

$$i\sigma C = -i\frac{m^2+\gamma^2}{4\sigma} ghC(1 - A_1) + \frac{m^2+\gamma^2}{4\sigma} ghCb_2, \tag{8}$$

or

$$-\frac{4\sigma^2}{c^2} = (m^2 + \gamma^2)(1 - a_1 + i(b_1 + b_2)), \tag{8^1}$$

where

$$A = a_1 - ib_1 = \frac{1}{h}\int_0^h \Phi_1 dz, \qquad b_2 = \frac{1}{h}\int_0^h \psi_2 dz. \tag{9}$$

As a first example we take the case:

$$\nu = \nu_0\left(1 - \frac{\alpha z}{h}\right).$$

We then get

$$\psi_2 = \frac{2\sigma h}{\nu_0}\int_0^z \frac{h-z}{h-\alpha z}\,dz = \frac{2\sigma h}{\nu_0\alpha}\left[z + \left(\frac{1}{\alpha} - 1\right)h\ln\left(1 - \frac{\alpha z}{h}\right)\right], \tag{10}$$

$$b_2 = \frac{1}{h}\int_0^h \psi_2\,dz = \frac{\sigma h^2}{\nu\alpha^2}\left[3\alpha - 2 - \frac{2(1-\alpha)^2}{\alpha}\ln(1-\alpha)\right], \tag{11}$$

To find P(z) and Q(z) we have to integrate the equation:

$$\frac{d}{dz}\left[\left(1 - \frac{\alpha z}{h}\right)\frac{dU}{dz}\right] - \frac{2i\sigma}{\nu_0}U = 0, \tag{12}$$

or if we put $\beta = \sqrt{\frac{\sigma}{\nu_0}}$, the equation will be identic with the one already integrated.

In the other case already treated $\nu = \nu_0 e^{-\frac{\alpha z}{h}}$, we get

$$\psi_2 = \frac{2\sigma}{\nu_0}\int_0^z e^{\frac{\alpha z}{h}}(h - z)\,dz = \frac{2\sigma h^2}{\nu\alpha^2}\left[(1+\alpha)(e^{\frac{\alpha z}{h}} - 1) - \frac{\alpha z}{h}e^{\frac{\alpha z}{h}}\right], \tag{13}$$

and

$$b_2 = \frac{1}{h}\int_0^h \psi_2\,dz = \frac{2\sigma h^2}{\nu\alpha^2}\left[\frac{2}{\alpha}(e^{\alpha} - 1) - (2+\alpha)\right]. \tag{14}$$

For P(z) and Q(z) we get the equation:

$$\frac{d}{dz}\left(e^{-\frac{\alpha z}{h}}\frac{dU}{dz}\right) - \frac{2i\sigma}{\nu_0}U = 0. \tag{15}$$

Or if we again put $\beta = \sqrt{\frac{\sigma}{\nu_0}}$, we get the equation already integrated.

14. Semi-convergent Expansions for H_1 and H_2.

The argument of the Cylinder functions is ϱx, where $\varrho = \frac{2\beta h}{\alpha}(1 - i)$ and $x = \sqrt{1 - \frac{\alpha z}{h}}$, or $e^{\frac{\alpha z}{2h}}$.

The range of x is from 1 to $\sqrt{1-\alpha}$ or $e^{\frac{\alpha}{2}}$, and for $\beta = \beta_1 = \sqrt{\frac{\sigma+\lambda}{2\nu_0}}$ the argument will in the most cases be comparatively great, and the calculation of the cylinder functions may, therefore, conveniently be done by means of the semi-convergent series for these functions.

Generally we have:[1])

$$H_1{}^p(re^{i\vartheta}) = \frac{e^{-r\sin\vartheta}}{\sqrt{\frac{1}{2}\pi r}}\, e^{i\left(r\cos\vartheta - \frac{2p+1}{4}\pi - \frac{\vartheta}{2}\right)} S_p\left(re^{i\left(\vartheta-\frac{\pi}{2}\right)}\right),$$

$$H_2{}^p(re^{i\vartheta}) = \frac{e^{r\sin\vartheta}}{\sqrt{\frac{1}{2}\pi r}}\, e^{-i\left(r\cos\vartheta - \frac{2p+1}{4}\pi + \frac{\vartheta}{2}\right)} S_p\left(re^{i\left(\vartheta+\frac{\pi}{2}\right)}\right), \tag{1}$$

$$S_p(x) = 1 + \sum_{n=1,2\ldots} \frac{(4p^2-1)(4p^2-3^2)\ldots(4p^2-(2n-1)^2)}{n!\,(8x)^n}. \tag{2}$$

In these formulae we have to put:

$$r = \frac{2\beta h}{\alpha} x\sqrt{2}, \quad \vartheta = -\frac{\pi}{4}.$$

We then get:

$$H_1{}^0(\varrho x) = \frac{e^{\frac{2\beta h x}{\alpha}(1+i) - \frac{i\pi}{8}}}{\sqrt{\frac{\pi\beta h}{\alpha} x\sqrt{2}}} S_0\left(-\frac{2\beta h x}{\alpha}\sqrt{2}\, e^{i\frac{\pi}{4}}\right),$$

$$H_2{}^0(\varrho x) = \frac{e^{-\frac{2\beta h x}{\alpha}(1+i) + \frac{3i\pi}{8}}}{\sqrt{\frac{\pi\beta h}{\alpha} x\sqrt{2}}} S_0\left(\frac{2\beta h x}{\alpha}\sqrt{2}\, e^{i\frac{\pi}{4}}\right), \tag{3}$$

$$H_1{}^1(\varrho x) = -i\frac{e^{\frac{2\beta h x}{\alpha}(1+i) - i\frac{\pi}{8}}}{\sqrt{\frac{\pi\beta h}{\alpha} x\sqrt{2}}} S_1\left(-\frac{2\beta h x}{\alpha}\sqrt{2}e^{\frac{i\pi}{4}}\right),$$

$$H_2{}^1(\varrho x) = i\frac{e^{\frac{-2\beta h x}{\alpha}(1+i) + \frac{3\pi i}{8}}}{\sqrt{\pi\frac{\beta h}{\alpha} x\sqrt{2}}} S_1\left(\frac{2\beta h x}{\alpha}\sqrt{2}\, e^{\frac{i\pi}{4}}\right). \tag{3}$$

For large values of the argument $S(x)$ may be approximately replaced by unity.

In the case $\nu = \nu_0 e^{-\frac{\alpha z}{h}}$ we then get:

$$\Phi_1 = \sqrt{x}\,\frac{\cosh\left[\frac{2\beta h}{\alpha}(x_1 - x)(1+i)\right]}{\cosh\left[\frac{2\beta h}{\alpha}(x_1-1)(1+i)\right]}, \tag{4}$$

and

$$A_1 = \frac{1-i}{2\beta_1 h}\tanh\left[\frac{2\beta h}{\alpha}(x_1-1)(1+i)\right]. \tag{5}$$

[1]) Jahnke und Emde: Funktionen-Tafeln.

We note the resemblance between this value for A and the value obtained under the assumption of constant eddy viscossity.

15. Numerical Examples.

We may now use the obtained formulae to calculate some examples which show how the distribution of the velocities depends on the variability of the eddy viscossity. We select the three cases

$$\nu = \nu_0 e^{-\frac{\alpha z}{h}} \qquad \nu = \text{const.} \qquad \nu = \nu_0 e^{\frac{\alpha z}{h}},$$

the mean value of ν in all cases being the same, about 220. α shall be so chosen, that in the first case ν decreases to half of its value at the bottom, and in the third case ν increases to the double of its value at the bottom. h is 50 m. and to simplify the calculations we chose $s = 1$. The velocities are adjusted to give the same mean energy in each case. In the case $\nu = \text{const.}$ we have furthermore assumed C = 10 cm.

In the following table we have grouped the calculated velocities and the orientations for comparisson, the first line of each group corresponds to $\nu = \nu_0 e^{-\frac{\alpha z}{h}}$, the second to $\nu = \text{const.}$ and the third to $\nu = \nu_0 e^{\frac{\alpha z}{h}}$.

d	0	5	10	20	30	50
	0	1.51	2.84	4.98	6.46	7.81
V	0	1.72	3.11	5.17	6.46	7.38
	0	1.97	3.49	5.38	6.50	7.12
	0	0.51	1.12	2.53	3.90	5.36
$-\alpha V$	0	0.64	1.34	2.77	3.96	5.01
	0	0.79	1.66	3.10	4.17	4.93
	0.262	0.336	0.395	0.507	0.604	0.686
$-\alpha$	0.304	0.374	0.432	0.535	0.631	0.679
	0.331	0.400	0.477	0.575	0.642	0.693
	+ 1°.5	− 2°.6	− 6°.9	− 14°.2	− 19°.1	− 21°.4
χ	+ 2°.3	− 2°.9	− 7°.6	− 14°.4	− 18°.4	− 20°.5
	+ 4°.9	− 3°.1	− 6°.7	− 13°.6	− 18°.4	− 17°.8
	− 24°.5	− 20°.8	− 16°.5	− 9°.3	− 4°.4	− 2°.0
τ	− 24°.8	− 19°.4	− 14°.4	− 7°.9	− 3°.9	− 1°.9
	− 24°.4	− 17°.5	− 12°.3	− 5°.9	− 2°.1	− 1°.5

Wee see that the configuration of the motion is on the whole the same: χ varying from 0 close at the bottom to about − 20° at the surface, and τ from − 25 to 0. The ratio of the axes, α, varies from about 0.7 at the surface to 0.3 at the bottom.

The velocities decrease in the same way from surface to bottom, but a closer inspection of the values reveals that the influence of the viscossity is greatest in the case

$$\nu = \nu_0 e^{-\frac{\alpha z}{h}} \quad \text{and least in the case} \quad \nu = \nu_0 e^{\frac{\alpha z}{h}}.$$

This is also seen from the exponent of damping p and the velocity of propagation which are in the three cases:

$$p = 0.585\,,\quad 0{,}568 \quad \text{and} \quad 0.511\,.$$
$$\varkappa = 38,\quad 38.9 \quad \text{and} \quad 41.6 \text{ m/sec}\,.$$

This is what we should expect as the influence of the viscossity on the motion is always greatest in the bottom-layer.

Now the laws which we have used above for the distribution of the eddy viscossity are not likely to hold in any case which we really meet with in nature. The eddy viscossity will generally depend on the hydrographic conditions in a way not hitherto fully explained. The calculations are never the less of interest, as they show how the distribution of velocity which we have found in the case of constant eddy viscossity, is modified when the eddy viscossity varies with the depth.

When the eddy viscossity does not show abrupt changes, we may get an approximate solution in the following way:

In the equation

$$\frac{d}{dz}\left(\nu\frac{d\Phi}{dz}\right) - i(\sigma+\lambda)\Phi = 0 \tag{1}$$

we introduce a new variable by

$$x = \sqrt{\bar{\nu}}\int_0^z \frac{dz}{\sqrt{\nu}}\,, \qquad \frac{1}{\sqrt{\bar{\nu}}} = \frac{1}{h}\int_0^h \frac{dz}{\sqrt{\nu}}\,, \tag{2}$$

the equation then takes the form

$$\frac{d^2\Phi}{dx^2} + \frac{1}{2}\frac{\frac{d\nu}{dz}}{\sqrt{\nu\bar{\nu}}}\frac{d\Phi}{dx} - i\frac{\sigma+\lambda}{\bar{\nu}}\Phi = 0\,,$$

or

$$\frac{d^2\Phi}{dx^2} + \frac{1}{2\nu}\frac{d\nu}{dx}\frac{d\Phi}{dx} - 2i\beta^2\Phi = 0\,. \tag{3}$$

We then put $\Phi = \sqrt[4]{\frac{\nu_0}{\nu}}\,\Phi_1$, where ν_0 is the value of ν for $x = 0$. The equation then takes the form:

$$\frac{d^2\Phi_1}{dx^2} - \left[2i\beta^2 + \frac{1}{4}\frac{\nu''}{\nu} - \frac{3}{16}\left(\frac{\nu'}{\nu}\right)^2\right]\Phi_1 = 0\,. \tag{4}$$

The expression $\frac{1}{4}\frac{\nu''}{\nu} - \frac{3}{16}\left(\frac{\nu'}{\nu}\right)^2$, or if we introduce z as an independent variable $\frac{1}{4\bar{\nu}}\left(\nu'' - \frac{1}{4}\frac{\nu'^2}{\nu}\right)$, is often a very small quantity, we may, therefore, replace it by its mean value, or omit it altogether. An approximate integral is then:

$$\Phi = \sqrt[4]{\frac{\nu_0}{\nu}}\,\frac{\cosh\beta(1+i)(h-x)}{\cosh\beta(1+i)h}, \tag{5}$$

And with sufficient accuracy we may also write:

$$A = \frac{1}{h}\int_0^h \Phi dz = \frac{1-i}{2\beta h}\sqrt{\frac{\nu_0}{\bar{\nu}}}\tanh\beta(1+i)h.$$

The method gives the exact solution in two cases, $\nu =$ const. and $\nu = \nu_0\left(1+\frac{\alpha z}{h}\right)^2$ where $\frac{1}{4\bar{\nu}}\left(\nu'' - \frac{1}{4}\frac{\nu'^2}{\nu}\right)$ is constant.

To show how the method may be used with advantage, we take the case already treated $\nu = \nu_0 e^{-\frac{\alpha z}{h}}$. We then have

$$x = \sqrt{\frac{\bar{\nu}}{\nu_0}}\int_0^z e^{\frac{\alpha z}{2h}}dz = \frac{2h}{\alpha}\sqrt{\frac{\bar{\nu}}{\nu_0}}\left(e^{\frac{\alpha z}{2h}}-1\right), \tag{6}$$

and

$$\frac{1}{4\bar{\nu}}\left(\nu'' - \frac{1}{4}\frac{\nu'^2}{\nu}\right) = \frac{3}{16}\frac{\nu_0}{\bar{\nu}}\frac{\alpha^2}{h^2}e^{-\frac{\alpha z}{h}} < \frac{3}{16}\frac{\nu_0}{\bar{\nu}}\frac{\alpha^2}{h^2}. \tag{7}$$

If we compare $2\beta^2$ and $\frac{3}{16}\frac{\nu_0}{\bar{\nu}}\frac{\alpha^2}{h^2}$ or $2\left(\frac{\beta h}{\alpha}\right)^2$ and $\frac{3}{16}\frac{\nu_0}{\bar{\nu}}$, we have in the calculated example $2\left(\frac{\beta h}{\alpha}\right)^2 = 50$ and $\frac{3}{16}\frac{\nu_0}{\bar{\nu}} = 0.26$. The omission of $\frac{1}{4\bar{\nu}}\left(\nu'' - \frac{1}{4\nu}\nu'^2\right)$ gives exactly the assymptotic expression which we have used above.

We see that apart from the factor $\sqrt[4]{\frac{\nu_0}{\nu}}$ the function Φ has the same form as in the case of constant eddy viscossity, and the effect of the variable viscossity is approximately as if the abscissa were elongated or contracted in the ratio

$$\frac{dx}{dz} = \sqrt{\frac{\bar{\nu}}{\nu}}.$$

The variation of Φ with z is therefore slower when $\nu > \bar{\nu}$ and more rapid when $\nu < \bar{\nu}$. This is what we have found in the preceding examples.

16. General Discussion of the Influence of Variable Viscossity when the Surface is Free.

It is interesting to see how far the conclusions regarding the distribution of velocity with depth when the eddy viscossity is constant, may hold good in the general case when ν is any continuous function of the depth which fulfils the condition $\nu > 0$.

As Φ and $A = \frac{1}{h}\int_0^h \Phi dz$ are complex quantities we put:

$$\Phi_1 = \varphi_1 - i\psi_1, \quad \Phi_2 = \varphi_2 - i\psi_2, \quad A_1 = a_1 - ib_1, \quad A_2 = a_2 - ib_2, \tag{1}$$

φ and ψ are then determined by the simultaneous differential-equations:

$$\frac{d}{dz}\left(k\frac{d\varphi}{dz}\right) = 2\beta^2\psi, \qquad \nu = \nu_0 k(z),$$
$$\frac{d}{dz}\left(k\frac{d\psi}{dz}\right) = -2\beta^2\varphi, \quad \beta = \sqrt{\frac{\sigma+\lambda}{2\nu_0}}, \tag{2}$$

with the boundary conditions:

$$\varphi = 1,\ \psi = 0,\ z = 0;\ \frac{d\varphi}{dz} = \frac{d\psi}{dz} = 0,\ z = h. \tag{3}$$

By a combination of these equations we get:

$$\int_z^h \left[\varphi\frac{d}{dz}\left(k\frac{d\varphi}{dz}\right) + \psi\frac{d}{dz}\left(k\frac{d\psi}{dz}\right)\right]dz = 0, \tag{4}$$

or

$$\varphi\varphi' + \psi\psi' = -\frac{1}{k}\int_z^h k(\varphi'^2 + \psi'^2)\,dz. \tag{4^1}$$

In the same manner we get:

$$\varphi\psi' - \varphi'\psi = \frac{2\beta^2}{k}\int_z^h (\varphi^2 + \psi^2)\,dz. \tag{5}$$

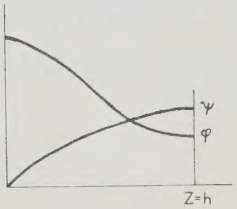
We therefore have:

$$\varphi\varphi' + \psi\psi' < 0, \quad \varphi\psi' - \varphi'\psi > 0. \tag{6}$$

These two conditions give some information about the variations of the functions φ and ψ.

For $z = 0$ we have $\varphi = 1$, $\psi = 0$ and, therefore, $\psi'(0) = 2\beta^2 ha > 0$, $\varphi'(0) = -2\beta^2 hb < 0$. At the surface we have always $\varphi' = 0\,\psi' = 0$. Now we may at first assume that we reach the value $z = h$ before φ becomes 0.

We then have

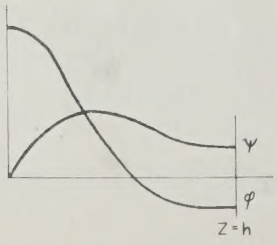


$$\varphi > 0, \quad \psi > 0, \quad \varphi' = 0, \quad \psi' = 0,$$
$$\varphi'' = 2\beta^2\psi, \quad \psi'' = -2\beta^2\varphi,$$

and we conclude that for $z = h$, φ has a minimum and ψ a maximum, this corresponds to the behaviour of the functions φ_2 and ψ_2 in the case calculated p. 27. Then we suppose that h is greater so that φ has a zero. For $\varphi = 0$ we must have

$$\psi\psi' < 0, \quad \varphi'\psi < 0 \quad \text{and as} \quad \psi > 0 \quad \text{we have} \quad \psi' < 0, \quad \varphi' < 0.$$

ψ has passed its maximum value. For the value of z which corresponds to maximum of ψ we have $\psi' = 0$, and, therefore, $\varphi\varphi' < 0$, $\varphi'\psi < 0$ and so $\varphi' < 0$, $\varphi > 0$. If we now suppose that φ is still negative for $z = h$, we have $\varphi < 0$, $\varphi' = 0$ and concequently



$$\psi\psi' = 0, \qquad \varphi\psi' = 0,$$
$$\varphi'' = 2\beta^2\psi, \qquad \psi'' = -2\beta^2\varphi,$$

as $\psi > 0$ we have $\varphi'' > 0$, and φ has a minimum,

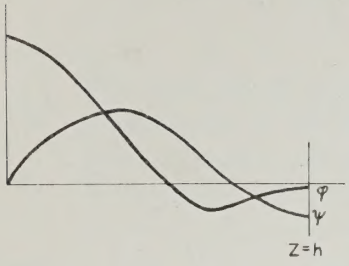
$$\varphi < 0, \quad \psi'' > 0, \quad \psi \text{ minimum.}$$

If h is still greater ψ may pass through the value zero. For $\psi = 0$ we have $\psi' < 0$

$$\varphi\varphi' < 0, \qquad \varphi\psi' > 0,$$

we, therefore, conclude $\varphi < 0$, $\varphi' > 0$, and φ has passed its minimum.

If φ and ψ are still negative until $z = h$, we have $\varphi'' < 0$, $\psi'' > 0$, and φ has a maximum, ψ a minimum for $z = h$.



Thus we may proceed and follow the configuration of the two curves. We see that a zero-point of one of the functions falls between two consecutive zero-points of the other, and that φ and ψ reach their extreme values before the nearest following zero of the other function. The character of the two functions may also be recognized if we write them in the form

$$\varphi = e^{-\varepsilon} \cos \delta \qquad \psi = e^{-\varepsilon} \sin \delta. \tag{7}$$

We then get

$$e^{-2\varepsilon} = \varphi^2 + \psi^2$$

and

$$e^{-2\varepsilon} \frac{d\varepsilon}{dz} = -\varphi\varphi' - \psi\psi' = \frac{1}{k} \int_z^h k(\varphi'^2 + \psi'^2) dz$$

and therefore

$$\frac{d\varepsilon}{dz} = \frac{e^{2\varepsilon}}{k} \int_z^h k(\varphi'^2 + \psi'^2) dz, \tag{8}$$

consequently $\frac{d\varepsilon}{dz}$ is always positive. For δ we have

$$\tan \delta = \frac{\psi}{\varphi}$$

and

$$\frac{d\delta}{dz} = \frac{\varphi\psi' - \varphi'\psi}{\varphi^2 + \psi^2} = \frac{2\beta^2 e^{2\varepsilon}}{k} \int_z^h e^{-2\varepsilon} dz \tag{9}$$

so that $\frac{d\delta}{dz} > 0$. δ increases from the bottom to the surface. For $z = h$ we find:

$$\frac{d\delta}{dz} = 0, \quad \frac{d\varepsilon}{dz} = 0, \quad \text{and} \quad \frac{d^2\delta}{dz^2} = -\frac{2\beta^2}{k},$$

$$\frac{d^2\varepsilon}{dz^2} = \frac{d^3\varepsilon}{dz^3} = 0, \quad \frac{d^4\varepsilon}{dz^4} = -\frac{8\beta^4}{k^2},$$

so that δ and ε have a maximum at the surface.

To compare φ_1 ψ_1 etc. with φ_2, ψ_2 we start with the differential-equations:

$$\begin{aligned} \frac{d}{dz}\left(k \frac{d\varphi_1}{dz}\right) &= 2\beta_1^2 \psi_1, \quad \frac{d}{dz}\left(k \frac{d\psi_1}{dz}\right) = -2\beta_1^2 \varphi_1, \\ \frac{d}{dz}\left(k \frac{d\varphi_2}{dz}\right) &= 2\beta_2^2 \psi_2, \quad \frac{d}{dz}\left(k \frac{d\psi_2}{dz}\right) = -2\beta_2^2 \varphi_2. \end{aligned} \tag{10}$$

Combining these equations, and integrating between zero and h, we obtain:

$$\int_0^h \left[\varphi_2 \frac{d}{dz}\left(k \frac{d\varphi_1}{dz}\right) - \varphi_1 \frac{d}{dz}\left(k \frac{d\varphi_2}{dz}\right)\right] dz = 2\beta_1^2 \int_0^h \varphi_2 \psi_1 dz - 2\beta_2^2 \int_0^h \varphi_1 \psi_2 dz,$$

or $$-\varphi_1'(0) + \varphi_2'(0) = 2\beta_1^2 \int_0^h \varphi_2 \psi_1 \, dz - 2\beta_2^2 \int_0^h \varphi_1 \psi_2 \, dz, \tag{11}$$

and in the same way:

$$0 = 2\beta_2^2 \int_0^h \varphi_2 \psi_1 \, dz - 2\beta_1^2 \int_0^h \varphi_1 \psi_2 \, dz. \tag{12}$$

Subtracting this from the preceding equation, we get:

$$-\varphi_1'(0) + \varphi_2'(0) = 2(\beta_1^2 - \beta_2^2) \int_0^h (\varphi_2 \psi_1 + \varphi_1 \psi_2) \, dz. \tag{13}$$

The integral on the right side is evidently positive and as $\beta_1 > \beta_2$, we conclude that:

$$-\varphi_1'(0) > \varphi_2'(0),$$

or $$\varepsilon_1'(0) > \varepsilon_2'(0),$$

ε_1 increases faster than ε_2 at the bottom.

Inserting the values of β_1 and β_2 in equation (12) we get

$$\int_0^h (\varphi_2 \psi_1 - \varphi_1 \psi_2) \, dz = s \int_0^h (\varphi_2 \psi_1 + \varphi_1 \psi_2) \, dz \tag{14}$$

or as we may write

$$\int_0^h e^{-\varepsilon_1 - \varepsilon_2} \sin(\delta_1 - \delta_2) = s \int_0^h e^{-\varepsilon_1 - \varepsilon_2} \sin(\delta_1 + \delta_2) \, dz. \tag{15}$$

From this equation we may conclude that $\delta_1 > \delta_2$.

From the two equations

$$\beta_2^2 \int_0^h \varphi_2 \frac{d}{dz}\left(k \frac{d\psi_1}{dz}\right) dz - \beta_1^2 \int_0^h \varphi_1 \frac{d}{dz}\left(k \frac{d\psi_2}{dz}\right) dz = 0 \tag{16}$$

$$\beta_2^2 \int_0^h \psi_2 \frac{d}{dz}\left(k \frac{d\varphi_1}{dz}\right) dz - \beta_1^2 \int_0^h \psi_1 \frac{d}{dz}\left(k \frac{d\varphi_2}{dz}\right) dz = 0 \tag{17}$$

we get:

$$\beta_1^2 \psi_2'(0) - \beta_2^2 \psi_1'(0) = 2(\beta_1^4 - \beta_2^4) \int_0^h \psi_1 \psi_2 \, dz, \tag{18}$$

$$\beta_1^2 \psi_1'(0) - \beta_2^2 \psi_2'(0) = 2(\beta_1^4 - \beta_2^4) \int_0^h \varphi_1 \varphi_2 \, dz. \tag{19}$$

These equations give

$$\psi_1'(0) - \psi_2'(0) = 2(\beta_1^2 - \beta_2^2) \int_0^h (\varphi_1 \varphi_2 - \psi_1 \psi_2) \, dz, \tag{20}$$

or as we may write

$$\delta_1'(0) - \delta_2'(0) = 2(\beta_1^2 - \beta_2^2) \int_0^h e^{-\varepsilon_1 - \varepsilon_2} \cos(\delta_1 + \delta_2)\, dz. \tag{21}$$

For small values of h at any rate we may conclude that $\delta_1'(0) > \delta_2'(0)$. This will generally be the case, according to the fact that $\delta_1 > \delta_2$.

From equation (18) we may also conclude $a_2 > a_1$.

To see how ε and δ depend on β, we may also consider the equations:

$$\frac{d\varepsilon}{dz} = \frac{1}{k} \int_z^h k \left[\left(\frac{d\delta}{dz}\right)^2 - \left(\frac{d\varepsilon}{dz}\right)^2 \right] dz, \tag{22}$$

$$\frac{d\delta}{dz} = \frac{2\beta^2}{k} e^{2\varepsilon} \int_z^h e^{-2\varepsilon}\, dz. \tag{23}$$

For $\frac{d\varepsilon}{dz}$ we also have the equation:

$$\frac{d\varepsilon}{dz} = \frac{e^{2\varepsilon}}{k} \int_z^h e^{-2\varepsilon} k \left[\left(\frac{d\delta}{dz}\right)^2 + \left(\frac{d\varepsilon}{dz}\right)^2 \right] dz. \tag{24}$$

By an application of Bonnet's mean value theorem, we get from (23)

$$\frac{d\delta}{dz} = \frac{2\beta^2}{k}(h_1 - z), \qquad z < h_1 < h.$$

Suppose that β increases, we shall see that $\frac{d\delta}{dz}$ and $\frac{d\varepsilon}{dz}$ must both increase. If this was not so, $\frac{d\varepsilon}{dz}$ might decrease. $e^{-2\varepsilon}$ then would decrease more slowly and $h_1 - z$ would increase. Then $\frac{d\delta}{dz}$ would be greater, and from (22) we see that $\frac{d\varepsilon}{dz}$ would increase against the assumption. Thus $\frac{d\delta}{dz}$ and $\frac{d\varepsilon}{dz}$ both increase with β and we may conclude that δ_1 increases faster than δ_2 and ε_1 faster than ε_2.

We see that the functions have the same character as in the case $\nu = \text{const.}$, so that the assumption of constant eddy viscossity is on the whole a permitted approximation when the surface is free.

For the angle ω which enter in the solution we have

$$\tan\omega = \frac{\psi}{1-\varphi},$$

$$\frac{d\omega}{dz} = \frac{\psi' - (\varphi\psi' - \varphi'\psi)}{(1-\varphi)^2 + \psi^2}, \tag{25}$$

or as

$$\psi' = \frac{2\beta^2}{k} \int_z^h \varphi\, dz,$$

and

$$\varphi\psi' - \varphi'\psi = \frac{2\beta^2}{k}\int_z^h (\varphi^2 + \psi^2)\,dz\,,$$

we have

$$\frac{d\omega}{dz} = \frac{2\beta^2}{k}\,\frac{\int_z^h (\varphi - \varphi^2 - \psi^2)\,dz}{(1-\varphi)^2 + \psi^2}\,.$$

Meanwhile

$$\int_0^h (\varphi - \varphi^2 - \psi^2)\,dz = 0\,,$$

so that we may also write

$$\frac{d\omega}{dz} = -\frac{2\beta^2}{k}\,\frac{\int_0^z (\varphi - \varphi^2 - \psi^2)\,dz}{(1-\varphi)^2 + \psi^2}\,. \tag{26}$$

Let us consider the integral:

$$\int_0^z (\varphi - \varphi^2 - \psi^2)\,dz = \int_0^z e^{-\varepsilon}\,(\cos\delta - e^{-\varepsilon})\,dz\,. \tag{27}$$

For small values of z, the function under the integral-sign is positive. We have:

$$\cos\delta = 1 - \frac{\delta^2}{2!} + \frac{\delta^4}{4!} - \ldots\ldots\,,$$

$$e^{-\varepsilon} = 1 - \frac{\varepsilon}{1!} + \frac{\varepsilon^2}{2!} - \frac{\varepsilon^3}{3!} + \ldots\ldots\,,$$

and therefore:

$$\cos\delta - e^{-\varepsilon} = \varepsilon - \frac{\delta^2 + \varepsilon^2}{2!} + \frac{\varepsilon^3}{3!} + \frac{\delta^4 - \varepsilon^4}{4!} + \ldots\ldots\,,$$

and as ε and δ increase from the value zero at the bottom, the element of the integral must be positive. For greater values of z, the element is negative and it does not change its sign before $\delta > \frac{3\pi}{2}$. If $\delta_h < \frac{3\pi}{2}$, we may therefore conclude that $\frac{d\omega}{dz}$ is negative from the bottom to the surface. If $\delta_h > \frac{3\pi}{2}$, $\frac{d\omega}{dz}$ might be positive but very small, and it is always negative near the bottom.

The limit for $z = 0$ is:

$$\left(\frac{d\omega}{dz}\right)_{z=0} = \frac{\beta^2\,\varphi'(0)}{\varphi'^2(0) + \psi'^2(0)} = -\frac{b}{2h(a^2+b^2)}\,. \tag{28}$$

At the surface we have $\frac{d\omega}{dz} = 0$ and:

$$\frac{d^2\omega}{dz^2} = \frac{2\beta^2}{k}\,\frac{\varphi^2 + \psi^2 - \varphi}{(1-\varphi)^2 + \psi^2} = \frac{\beta^2}{k}\,\frac{e^{-\varepsilon} - \cos\delta}{\cosh\varepsilon - \cos\delta}\,. \tag{29}$$

If $\frac{\pi}{2} < \delta_h < \frac{3\pi}{2}$ we have certainly $\frac{d^2\omega}{dz^2} > 0$ and ω has a minimum at the surface.

For the phase-difference between high water and max. current we have found:

$$\tau = -\frac{\omega_2 + \omega_2 - \gamma_1 - \gamma_2}{2}. \tag{30}$$

and therefore:

$$\frac{d\tau}{dz} = -\frac{d\omega_1}{dz} - \frac{d\omega_2}{dz}. \tag{31}$$

As τ is negative at the bottom and $\frac{d\omega}{dz} < 0$ we have $\frac{d\tau}{dz} > 0$ and the numerical value of τ will decrease. The phase-difference is greatest at the bottom and decreases upwards, this is what we have found in all the examples.

For the angle χ it is not so easy to come to a definite conclusion. In one case however the discussion is more easy. This is when $s = 1$. We then have $\frac{d\omega_2}{dz} = 0$, and consequently $\frac{d\chi}{dz} = \frac{d\omega_1}{dz}$, χ decreases as ω_1. The ellipse is therefore deflected to the right and this deflection increases from the bottom and upwards. We also know that when $\frac{d\nu}{dz}$ is only small, Φ is approximately:

$$\sqrt[4]{\frac{\nu_0}{\nu}}\,\frac{\cosh\beta(x_1 - x)(1+i)}{\cosh\beta(1+i)\,x_1},$$

and tan ω, has the same form as when ν is constant. In this case we know that ω_1 decreases faster than ω_2 so that χ varies as when ν is constant.

17. The Energy of the Tidal Wave.

The potential energy of the tidal wave due to the elevation or depression of the surface above or below the mean level is per unit area:

$$g\varrho \int_h^{h+\zeta} (z - h)\,dz = \frac{1}{2} g\varrho\zeta^2, \tag{1}$$

or

$$E_p = \frac{1}{2} g\varrho C^2 \cos^2\sigma\left(t - \frac{x}{z}\right). \tag{2}$$

and the mean potential energy during one period is:

$$\bar{E}_p = \frac{1}{T}\int_0^T E_p\,dt = \frac{1}{4} g\varrho C^2. \tag{3}$$

The kinetic energy of a water column extending from the bottom to the surface is per unit area

$$\frac{1}{2}\varrho\int_0^h (u^2 + v^2)\,dz. \tag{4}$$

In case of no eddy viscossity we have:

$$u = \sqrt{\frac{g}{h}}\, C \frac{1}{\sqrt{1-\alpha^2}} \cos\sigma\left(t - \frac{x}{\varkappa}\right),$$
$$v = -\sqrt{\frac{g}{h}}\, C \frac{\alpha}{\sqrt{1-\alpha^2}} \sin\sigma\left(t - \frac{x}{\varkappa}\right), \tag{5}$$

and we get:

$$E_k = \frac{1}{2} g\varrho C^2 \frac{1}{1-\alpha^2}\left[\cos^2\sigma\left(t - \frac{x}{\varkappa}\right) + \alpha^2 \sin^2\sigma\left(t - \frac{x}{\varkappa}\right)\right],$$

or

$$E_k = \frac{1}{2} g\varrho C^2 \left[\frac{\alpha^2}{1-\alpha^2} + \cos^2\sigma\left(t - \frac{x}{\varkappa}\right)\right]. \tag{6}$$

We see that

$$E_k = E_p + \frac{1}{2} g\varrho C^2 \frac{\alpha^2}{1-\alpha^2}. \tag{7}$$

The kinetic energy consists of a constant part and a variable part which is equal to the potential energy. The variable part of the kinetic energy is independent of α or θ.

The mean kinetic energy during one period is

$$\bar{E}_k = \frac{1}{T}\int_0^T E_k\, dt = \frac{1}{4} g\varrho C^2 \frac{1+\alpha^2}{1-\alpha^2} \tag{8}$$

and the total energy is

$$\bar{E}_T = \bar{E}_k + \bar{E}_p = \frac{1}{2} g\varrho C^2 \frac{1}{1-\alpha^2}. \tag{9}$$

When the eddy viscossity is taken in account the potential energy is

$$E_p = \frac{1}{2} g\varrho C^2 \cos^2\sigma\left(t - \frac{x}{\varkappa}\right) e^{-2\mu x - 2my}. \tag{10}$$

and the kinetic energy is

$$E_k = \frac{1}{2}\varrho \int_0^h (u^2 + v^2)\, dz. \tag{11}$$

In this case we have for u and v:

$$u = \sqrt{\frac{g}{h}}\, \frac{C}{2}\left[M\cos\sigma\left(t - \frac{x}{\varkappa}\right) + N\sin\sigma\left(t - \frac{x}{\varkappa}\right)\right] e^{-\mu x - my},$$
$$v = \sqrt{\frac{g}{h}}\, \frac{C}{2}\left[P\cos\sigma\left(t - \frac{x}{\varkappa}\right) + Q\sin\sigma\left(t - \frac{x}{\varkappa}\right)\right] e^{-\mu x - my}. \tag{12}$$

We may without loss of generality put $x = 0$ and $y = 0$, and then we have:

$$E_z = \frac{1}{16} g\varrho C^2 \frac{1}{h}\int_0^h dz\,[(M^2 + N^2 + P^2 + Q^2) + (M^2 + P^2 - N^2 - Q^2)\cos 2\sigma t + 2(MN + PQ)\sin 2\sigma t]. \tag{13}$$

For M, N etc. we have the expressions:

$$\begin{aligned} M &= A(1-\varphi_1) + A_1\psi_1 + B(1-\varphi_2) + B_1\psi_2, \\ N &= -A\psi_1 + A_1(1-\varphi_1) - B\psi_2 + B_1(1-\varphi_2), \\ P &= A\psi_1 - A_1(1-\varphi_1) - B\psi_2 + B_1(1-\varphi_2), \\ Q &= A(1-\varphi_1) + A_1\psi_1 - B(1-\varphi_2) - B_1\psi_2. \end{aligned} \tag{14}$$

We, therefore, get:

$$\begin{aligned} M^2 + N^2 + P^2 + Q^2 &= 2(A^2 + A_1{}^2)[(1-\varphi_1)^2 + \psi_1{}^2] \\ &\quad + 2(B^2 + B_1{}^2)[(1-\varphi_2)^2 + \psi_2{}^2], \\ M^2 + P^2 - N^2 - Q^2 &= 4(AB - A_1B_1)[(1-\varphi_1)(1-\varphi_2) - \psi_1\psi_2] \\ &\quad + 4(AB_1 + A_1B)[\psi_1(1-\varphi_2) + \psi_2(1-\varphi_1)], \\ 2(MN + PQ) &= 4(AB_1 + A_1B)[(1-\varphi_1)(1-\varphi_2) - \psi_1\psi_2] \\ &\quad - 4(AB - A_1B_1)[\psi_1(1-\varphi_2) + \psi_2(1-\varphi_1)]. \end{aligned} \tag{15}$$

We then have to evaluate the following integrals:

$$\int_0^h [(1-\varphi)^2 + \psi^2]\,dz, \qquad \int_0^h (1 - \varphi_1 - \varphi_2 + \varphi_1\varphi_2 - \psi_1\psi_2)dz$$

$$\int_0^h (\psi_1 + \psi_2 - \varphi_1\psi_2 - \varphi_2\psi_1)\,dz.$$

φ and ψ satisfy the differential-equations:

$$\frac{d}{dz}\left(k\frac{d\varphi}{dz}\right) = 2\beta^2\psi, \quad \frac{d}{dz}\left(k\frac{d\psi}{dz}\right) = -2\beta^2\varphi, \tag{16}$$

with the boundary-conditions:

$$\varphi = 1, \quad \psi = 0 \quad \text{for} \quad z = 0$$

$$\frac{d\varphi}{dz} = f(1-\varphi), \quad \frac{d\psi}{dz} = -f\psi, \quad \text{for } z = h.$$

Multiplying the first by ψ and the second by $-\varphi$ and integrating between 0 and h, we find:

$$\begin{aligned} 2\beta^2 \int_0^h (\varphi^2 + \psi^2)\,dz &= \int_0^h \left[\psi\frac{d}{dz}\left(k\frac{d\varphi}{dz}\right) - \varphi\frac{d}{dz}\left(k\frac{d\psi}{dz}\right)\right]dz \\ &= \left[\psi k\varphi' - \varphi k\psi'\right]_0^h = \psi'(0) + f(k\psi)_h, \end{aligned} \tag{17}$$

and as we have:

$$2\beta^2 \int_0^h \varphi\,dz = -\int_0^h \frac{d}{dz}\left(k\frac{d\psi}{dz}\right)dz = \psi'(0) + f(k\psi)_h = 2\beta^2 ha, \tag{18}$$

we obtain

$$\frac{1}{h}\int_0^h (\varphi^2 + \psi^2)\, dz = \frac{1}{h}\int_0^h \varphi\, dz = a\,, \tag{19}$$

and, therefore,

$$\frac{1}{h}\int_0^h [(1-\varphi)^2 + \psi^2]\, dz = 1 - a\,. \tag{20}$$

To evaluate the second integral, we make the following combinations:

$$2\beta_1^2 \int_0^h \varphi_1 \varphi_2\, dz + 2\beta_2^2 \int_0^h \psi_1 \psi_2\, dz = -\int_0^h \left[\varphi_2 \frac{d}{dz}\left(k\frac{d\psi_1}{dz}\right) - \psi_1 \frac{d}{dz}\left(k\frac{d\varphi_2}{dz}\right)\right] dz$$
$$= -\left[\varphi_2 k\psi_1' - \psi_1 k\varphi_2'\right]_0^h = \psi_1'(0) + f(k\psi_1)_h = 2\beta_1^2 h a_1\,, \tag{21}$$

and

$$2\beta_2^2 \int_0^h \varphi_1 \varphi_2\, dz + 2\beta_1^2 \int_0^h \psi_1 \psi_2\, dz = -\int_0^h \left[\varphi_1 \frac{d}{dz}\left(k\frac{d\psi_2}{dz}\right) - \psi_2 \frac{d}{dz}\left(k\frac{d\varphi_1}{dz}\right)\right] dz$$
$$= -\left[\varphi_1 k\psi_2' - \psi_2 k\varphi_1'\right]_0^h = \psi_2'(0) + f(k\psi_2)_h = 2\beta_2^2 h a_2\,. \tag{22}$$

Subtracting this equation from the preceding, we get:

$$\frac{1}{h}\int_0^h (\varphi_1 \varphi_2 - \psi_1 \psi_2)\, dz = \frac{\beta_1^2 a_1 - \beta_2^2 a_2}{\beta_1^2 - \beta_2^2}\,. \tag{23}$$

In the same manner we find:

$$2\beta_1^2 \int_0^h \varphi_1 \psi_2\, dz - 2\beta_2^2 \int_0^h \varphi_2 \psi_1\, dz = -\int_0^h \left[\psi_2 \frac{d}{dz}\left(k\frac{d\psi_1}{dz} - \psi_1 \frac{d}{dz}\left(k\frac{d\psi_2}{dz}\right)\right] dz$$
$$= \left[\psi_2 k\psi_1' - \psi_1 k\psi_2'\right]_0^h = 0\,, \tag{24}$$

and

$$2\beta_2^2 \int_0^h \varphi_1 \psi_2\, dz - 2\beta_1^2 \int_0^h \varphi_2 \psi_1\, dz = +\int_0^h \left[\varphi_1 \frac{d}{dz}\left(k\frac{d\varphi_2}{dz}\right) - \varphi_2 \frac{d}{dz}\left(k\frac{d\varphi_1}{dz}\right)\right] dz$$
$$= \left[\varphi_1 k\varphi_2' - \varphi_2 k\varphi_1'\right]_0^h = \varphi'(0) - \varphi_2'(0) + f(k\varphi_1)_h - f(k\varphi_2)_h\,. \tag{25}$$

From these equations we obtain:

$$\frac{1}{h}\int_0^h \varphi_1 \psi_2 + \psi_1 \varphi_2 = \frac{\beta_1^2 b_1 - \beta_2^2 b_2}{\beta_1^2 - \beta_2^2}\,. \tag{26}$$

Introducing these expressions in the integrals which we had to evaluate, we find:

$$\frac{1}{h}\int_0^h (1-\varphi_1-\varphi_2+\varphi_1\varphi_2-\psi_1\psi_2)dz = 1-\frac{1-\frac{1}{s}}{2}a_1-\frac{1+\frac{1}{s}}{2}a_2 = 1-\delta_1\,,$$

$$\frac{1}{h}\int_0^h (\psi_1+\psi_2-\psi_1\varphi_2-\psi_2\varphi_1)\,dz = \frac{1-\frac{1}{s}}{2}b_1+\frac{1+\frac{1}{s}}{2}b_2 = \varepsilon_1\,. \tag{27}$$

The notation is so chosen because of the analogy with the quantities $1-\delta$ and ε, where s is replaced by $\frac{1}{s}$.

The coefficients of the expression for E_k admit some simplifications. We have

$$AB - A_1B_1 = \frac{(\vartheta+s\theta)(\vartheta-s\theta)}{1-s^2} - \frac{p^2\vartheta^2}{1-s^2} = \frac{\vartheta^2(1-p^2)-s^2\theta^2}{1-s^2}\,, \tag{28}$$

but from the equation of continuity we know that this is equal to

$$k_1 = \frac{1-\delta}{(1-\delta)^2+\varepsilon^2}\,.$$

In the same manner we have

$$AB_1 + A_1B = \frac{2p\vartheta^2}{1-s^2} = k_2 = \frac{\varepsilon}{(1-\delta)^2+\varepsilon^2}\,. \tag{29}$$

For E_k we now obtain;

$$E_k = \frac{1}{4}g\varrho C^2\Big[(A^2+A_1{}^2)\frac{1-a_1}{2}+(B^2+B_1{}^2)\frac{1-a_2}{2}+[k_1(1-\delta_1)+k_2\varepsilon_1]\cos 2\sigma t + [k_2(1-\delta_1)-k_1\varepsilon_1]\sin 2\sigma t\Big] \tag{30}$$

which may be written:

$$E_k = \frac{1}{4}g\varrho C^2\Big[(A^2+A_1{}^2)\frac{1-a_1}{2}+(B^2+B_1{}^2)\frac{1-a_2}{2}-R+2R\cos^2(\sigma t-\tau_1)\Big]. \tag{30^1}$$

Here

$$R = \sqrt{\frac{(1-\delta_1)^2+\varepsilon_1{}^2}{(1-\delta)^2+\varepsilon^2}}\,, \tag{31}$$

and

$$\tan 2\tau_1 = \frac{k_2(1-\delta_1)-k_1\varepsilon_1}{k_1(1-\delta_1)+k_2\varepsilon_1} = \frac{\frac{\varepsilon}{1-\delta}-\frac{\varepsilon_1}{1-\delta_1}}{1-\frac{\varepsilon}{1-\delta}\frac{\varepsilon_1}{1-\delta_1}}\,. \tag{32}$$

We see that E_k consists of a constant part and a variable part as in the case of no resistance. The variable part is independent of θ, but is not equal to the potential energy. The phase is different and the amplitude bears to the amplitude of the potential energy the ratio

$$\sqrt{\frac{(1-\delta_1)^2+\varepsilon_1{}^2}{(1-\delta)^2+\varepsilon^2}}\,.$$

If we take the examples calculated pag. 27 and pag. 28, we find that this ratio is 0.98 and 0.97 resp. The variable part of the kinetic energy is smaller than the potential energy.

The mean kinetic energy is

$$E_k = \frac{1}{4} g \varrho C^2 \left((A^2 + A_1{}^2) \frac{1 - a_1}{2} + (B^2 + B_1{}^2) \frac{1 - a_2}{2} \right). \tag{33}$$

If $s > 1$ this expression is unaltered. In the case $s = 1$ we have to find the limit $\frac{1 - a_2}{(1 - s)^2}$.

More simply we may start with the equations for this case, and if $\nu = \text{const.}$ we find

$$E_k = \frac{1}{8} g \varrho C^2 \left[(A^2 + A_1{}^2)(1 - a_1) + (B^2 + B_1{}^2) \frac{\sigma^2 h^4}{\nu^2} \left(\frac{1}{30} + \frac{3 + fh}{6(1 + fh)^2} \right) \right]. \tag{34}$$

The dissipation of energy which takes place may be calculated by means of the dissipation-function of Rayleigh.

The mean dissipation during one periode in a water-column of unit base is

$$2F = \frac{\varrho}{T} \int_0^T dt \int_0^h \nu \left[\left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{dz} \right)^2 \right] dz \tag{35}$$

or if we insert the values of $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$, and integrate with respect to t

$$2F = \frac{1}{8} g \varrho C^2 \frac{1}{h} \int_0^h \nu \left[\left(\frac{dM}{dz} \right)^2 + \left(\frac{dN}{dz} \right)^2 + \left(\frac{dP}{dz} \right)^2 + \left(\frac{dQ}{dz} \right)^2 \right] dz. \tag{36}$$

If we again insert the values of $\frac{dM}{dz}$ etc., the above expression will be

$$2F = \frac{1}{4} g \varrho C^2 \frac{\nu_0}{h} \int_0^h \left[(A^2 + A_1{}^2)(k(z)(\varphi'_1{}^2 + \psi'_1{}^2)) + (B^2 + B_1{}^2)(k(z)(\varphi'_2{}^2 + \psi'_2{}^2)) \right] dz. \tag{37}$$

To perform the integrations we start with the equations (**16**, 10) and get

$$0 = \int_0^h \left[\varphi \frac{d}{dz} \left(k \frac{d\varphi}{dz} \right) + \psi \frac{d}{dz} \left(k \frac{d\psi}{dz} \right) \right] dz = \left[k (\varphi\varphi' + \psi\psi') \right]_0^h - \int_0^h k (\varphi'^2 + \psi'^2) \, dz \tag{38}$$

and, therefore,

$$\int_0^h k(\varphi'^2 + \psi'^2) \, dz = \left[k (\varphi\varphi' + \psi\psi') \right]_0^h = fk(1 - \varphi)_h - \varphi'(0) - fk \left[(1 - \varphi)^2 + \psi^2 \right]_h. \tag{39}$$

As we have

$$b = \frac{1}{h} \int_0^h \psi \, dz = \frac{1}{2\beta^2 h} \left(k\varphi' \right)_0^h = \frac{fk(1 - \varphi)_h - \varphi'(0)}{2\beta^2 h}, \tag{40}$$

the formula for 2F then will be

$$2F = \frac{1}{4} g\varrho C^2\sigma \left[(A^2 + A_1{}^2)(1+s)\left[b_1 - \frac{fh}{2\beta_1{}^2 h^2} k(h)((1-\varphi_1)^2 + \psi_1{}^2)_h\right] \right. \tag{41}$$

$$\left. + (B^2 + B_1{}^2)(1-s)\left[b_2 - \frac{fh}{2\beta^2 h^2} k(h)((1-\varphi_2)^2 + \psi_2{}^2)_h\right]\right].$$

In the case $f = 0$ we simply get

$$2F = \frac{1}{4} g\varrho C^2\sigma[(A^2 + A_1{}^2)(1+s)b_1 + (B^2 + B_1{}^2)(1-s)b_2]. \tag{42}$$

The total energy is

$$E_T = E_p + E_k = \frac{1}{4} g\varrho C^2\left(1 + \frac{A^2 + A_1{}^2}{2}(1-a_1) + \frac{B^2 + B_1{}^2}{2}(1-a_2)\right) e^{-2\mu x - 2my} \tag{43}$$

and the dissipation

$$2F = \frac{1}{4} g\varrho C^2\sigma[(A^2 + A_1{}^2)(1+s)b_1 + (B^2 + B_1{}^2)(1-s)b_2] e^{-2\mu x - 2my}. \tag{44}$$

The apparent velocity of propagation is $\varkappa = \frac{c}{\vartheta}$ and generally greater than c. The decrease of total energy in the direction of propagation is

$$-\frac{dE_T}{dx} = 2\mu E_T . \tag{45}$$

As the apparent velocity with which the wave propagates, is $\varkappa$, one might conclude that $2\mu E_T \varkappa = 2F$ or as $\mu\varkappa = \sigma p$

$$p\left(1 + (A^2 + A_1{}^2)\frac{1-a_1}{2} + (B^2 + B_1{}^2)\frac{1-a_2}{2}\right)$$
$$= (A^2 + A_1{}^2)(1+s)b_1 + (B^2 + B_1{}^2)(1-s)b_2 . \tag{46}$$

However, this equation is only correct if $s = 0$. It, therefore, seems as if these waves violate the principle of the constancy of energy, but this apparent contradiction is due to the fact that the propagation of an arbitrary impulse is not affected by the rotation and the velocity of propagation is $c = \sqrt{gh}$.

If, therefore, an impulse propagates, the front of the wave proceeds with the velocity c, but the motion will not come to rest when the wave has passed, it leaves a sort of residuum, such as Poincaré has found in the case of the equation

$$\frac{\partial^2 u}{\partial t^2} + \eta\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}. \tag{47}$$

In one simple case it is easy to prove this. Suppose that the motion is independent of y and that we do not take the viscossity in account. The equations of motion are

$$\frac{\partial u}{\partial t} - \lambda v = -g\frac{\partial \zeta}{\partial x}$$

$$\frac{\partial v}{\partial t} + \lambda u = 0 \tag{48}$$

$$\frac{\partial \zeta}{\partial t} = -h\frac{\partial u}{\partial x}.$$

Eliminating v and ζ, we obtain the equation

$$\frac{\partial^2 u}{\partial t^2} + \lambda^2 u = c^2 \frac{\partial^2 u}{\partial x^2}\,, \quad c^2 = gh\,. \tag{49}$$

If we put

$$u = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2}\int_0^t \left[f(x+c\tau)+f(x-c\tau)\right] \frac{\partial}{\partial t} \mathrm{J}_0 \left(\lambda\sqrt{t^2-\tau^2}\right) d\tau$$
$$+ \frac{1}{2}\int_0^t \left[\varphi(x+c\tau)+\varphi(x-c\tau)\right] \mathrm{J}_0 \left(\lambda\sqrt{t^2-\tau^2}\right) d\tau \tag{50}$$

where J_0 is Bessel's function of zero order, we have a solution which satisfies the conditions of Cauchy

$$u = f(x), \qquad \frac{\partial u}{\partial t} = \varphi(x) \text{ for } t = 0\,.$$

Suppose that $f(x)$ and $\varphi(x)$ are zero everywhere outside the interval $0 < x < x_1$.

If $x > x_1$, $u \equiv 0$ for $t < \dfrac{x-x_1}{c}$, then we have for $\dfrac{x}{c} > t > \dfrac{x-x_1}{c}$,

$$u = \frac{1}{2} f(x-ct) + \frac{1}{2}\int_{\frac{x-x_1}{c}}^t f(x-c\tau) \frac{\partial \mathrm{J}_0}{\partial t}\, d\tau + \frac{1}{2}\int_{\frac{x-x_1}{c}}^t \varphi(x-c\tau)\, \mathrm{J}_0\, d\tau, \tag{51}$$

and for $t > \dfrac{x}{c}$, we have:

$$u = \frac{1}{2}\int_{\frac{x-x_1}{c}}^{\frac{x}{c}} f(x-c\tau) \frac{\partial \mathrm{J}_0}{\partial t}\, d\tau + \frac{1}{2}\int_{\frac{x-x^1}{c}}^{\frac{x}{c}} \varphi(x-c\tau)\, \mathrm{J}_0\, d\tau\,. \tag{52}$$

We see that the wave has a definite front, but leaves a sort of residuum behind.

The wave which we consider is a superposition of such impulses, and we can, therefore, understand that the velocity with which the energy is propagated is not $\varkappa$, and that the distribution of energy in space does not depend on the dissipation only. However, if we wish to calculate the quantity of energy which is transformed to heat by means of the viscosity, we must use the dissipation-function.

18. Reflection of the Tidal Wave.

The reflection of tidal waves is generally very irregular, and it is therefore extremely difficult to obtain a theory which should take account of the changes in the tidal motion due to reflection, but in one simple case we can get an expression for the superposition of an incident and a reflected wave. This is when the motion is independent of y, and the reflection takes place at $x = l$.

For the incident wave we have

$$u_1 = \sqrt{\frac{g}{h}}\frac{C}{2}\left[M\cos\sigma\left(t-\frac{x}{\varkappa}\right)+N\sin\sigma\left(t-\frac{x}{\varkappa}\right)\right]e^{-\mu x}$$
$$v_1 = \sqrt{\frac{g}{h}}\frac{C}{2}\left[P\cos\sigma\left(t-\frac{x}{\varkappa}\right)+Q\sin\sigma\left(t-\frac{x}{\varkappa}\right)\right]e^{-\mu x}, \quad (1)$$

and for the reflected wave we have

$$u_2 = -\sqrt{\frac{g}{h}}\frac{C}{2}\left[M\cos\sigma\left(t-\frac{2l-x}{\varkappa}\right)+N\sin\sigma\left(t-\frac{2l-x}{\varkappa}\right)\right]e^{-\mu(2l-x)}$$
$$v_2 = -\sqrt{\frac{g}{h}}\frac{C}{2}\left[P\cos\sigma\left(t-\frac{2l-x}{\varkappa}\right)+Q\sin\sigma\left(t-\frac{2l-x}{\varkappa}\right)e^{-\mu(2l-x)}\right]. \quad (2)$$

The superposition of these waves gives for the motion

$$u = u_1 + u_2, \quad v = v_1 + v_2. \quad (3)$$

For ζ we have $\zeta_1 = Ce^{-\mu x}\cos\sigma\left(t-\frac{x}{\varkappa}\right)$ and $\zeta_2 = Ce^{-\mu(2l-x)}\cos\sigma\left(t-\frac{2l-x}{\varkappa}\right)$

$$\zeta = C\left[e^{-\mu x}\cos\sigma\left(t-\frac{x}{\varkappa}\right)+e^{-\mu(2l-x)}\cos\sigma\left(t-\frac{2l-x}{\varkappa}\right)\right]. \quad (4)$$

It is easily seen that the solution gives $u = 0$ and $v = 0$ for $x = l$.

Let us put

$$u = \sqrt{\frac{g}{h}}\frac{C}{2}(M_1\cos\sigma t + N_1\sin\sigma t),$$
$$v = \sqrt{\frac{g}{h}}\frac{C}{2}(P_1\cos\sigma t + Q_1\sin\sigma t). \quad (5)$$

For M_1, N_1, etc. we then have the following expressions

$$M_1 = M\left(e^{-\mu x}\cos\frac{\sigma x}{\varkappa}-e^{-\mu(2l-x)}\cos\frac{\sigma(2l-x)}{\varkappa}\right)-N\left(e^{-\mu x}\sin\frac{\sigma x}{\varkappa}-e^{-\mu(2l-x)}\sin\frac{\sigma(2l-x)}{\varkappa}\right)$$
$$N_1 = M\left(e^{-\mu x}\sin\frac{\sigma x}{\varkappa}-e^{-\mu(2l-x)}\sin\frac{\sigma(2l-x)}{\varkappa}\right)+N\left(e^{-\mu x}\cos\frac{\sigma x}{\varkappa}-e^{-\mu(2l-x)}\cos\frac{\sigma(2l-x)}{\varkappa}\right) \quad (6)$$

To get the expressions for P_1 and Q_1 we only have to replace M and N by P and Q respectively. For simplicity we may write

$$M_1 = M\alpha - N\beta, \quad N_1 = M\beta + N\alpha, \quad P_1 = P\alpha - Q\beta, \quad Q_1 = P\beta + Q\alpha. \quad (7)$$

The axes of the ellipse is given by the formulae

$$2V_1 = \sqrt{(M_1+Q_1)^2+(N_1-P_1)^2}+\sqrt{(M_1-Q_1)^2+(N_1+P_1)^2}$$
$$2\alpha V_1 = \sqrt{(M_1+Q_1)^2+(N_1-P_1)^2}-\sqrt{(M_1-Q_1)^2+(N_1+P_1)^2}. \quad (8)$$

That α here has another meaning will not cause any confusion.

We find

$$2V_1 = \left(\sqrt{(M+Q)^2+(N-P)^2}+\sqrt{(M-Q)^2+(N-P)^2}\right)\sqrt{\alpha^2+\beta^2} = 2V\sqrt{\alpha^2+\beta^2} \quad (9)$$

and in the same manner

$$\alpha V_1 = \alpha V \sqrt{\alpha^2 + \beta^2}$$

where

$$\sqrt{\alpha^2 + \beta^2} = e^{-\mu l} \sqrt{2\left[\cosh 2\mu(l-x) - \cos\frac{2\sigma}{\varkappa}(l-x)\right]}\,. \tag{10}$$

For the surface-elevation we may write

$$\zeta_1 = C_1 \cos(\sigma t - \tau_0) \tag{11}$$

where

$$C_1 = C e^{-\mu l} \sqrt{2\left[\cosh 2\mu(l-x) + \cos\frac{2\sigma}{\varkappa}(l-x)\right]} \tag{12}$$

and

$$\begin{aligned}\tan\tau_0 &= \frac{e^{-\mu x}\sin\frac{\sigma x}{\varkappa} + e^{-\mu(2l-x)}\sin\sigma\frac{(2l-x)}{\varkappa}}{e^{-\mu x}\cos\frac{\sigma x}{\varkappa} + e^{-\mu(2l-x)}\cos\sigma\frac{(2l-x)}{\varkappa}} \\ &= \frac{\tan\frac{\sigma x}{\varkappa} - \tan\frac{\sigma(l-x)}{\varkappa}\tanh\mu(l-x)}{1 + \tan\frac{\sigma l}{\varkappa}\tan\frac{\sigma(l-x)}{\varkappa}\tanh\mu(l-x)}\end{aligned} \tag{13}$$

or

$$\tau_0 = \frac{\sigma l}{\varkappa} - \vartheta_0\,,\ \tan\vartheta_0 = \tan\sigma\frac{(l-x)}{\varkappa}\tanh\mu(l-x)\,. \tag{14}$$

To calculate the time of maximum current we have the formula

$$\tan 2\tau_1 = \frac{2(M_1N_1 + P_1Q_1)}{M_1^2 - N_1^2 + P_1^2 - Q_1^2} \tag{15}$$

or as

$$M_1N_1 + P_1Q_1 = (M^2 - N^2 + P^2 - Q^2)\alpha\beta + (MN + PQ)(\alpha^2 - \beta^2)$$
$$M_1^2 - N_1^2 + P_1^2 - Q_1^2 = (M^2 - N^2 + P^2 - Q^2)(\alpha^2 - \beta^2) - 4(MN + PQ)\alpha\beta\,,$$

we get

$$\tan 2\tau_1 = \frac{\dfrac{2(MN+PQ)}{M^2 - N^2 + P^2 - Q^2} + \dfrac{2\frac{\beta}{\alpha}}{1 - \frac{\beta^2}{\alpha^2}}}{1 - \dfrac{2(MN+PQ)}{M^2 - N^2 + P^2 - Q^2}\dfrac{2\frac{\beta}{\alpha}}{1 - \frac{\beta^2}{\alpha^2}}}\,. \tag{16}$$

As we for the incident wave at $x = 0$ have

$$\tan 2\tau = \frac{2(MN + PQ)}{M^2 - N^2 + P^2 - Q^2}\,, \tag{17}$$

we get

$$\tan 2\tau_1 = \tan(2\tau + 2\psi)\,, \tag{18}$$

where

$$\tan\psi = \frac{\beta}{\alpha},$$

or

$$\tan\psi = \frac{e^{-\mu x}\sin\frac{\sigma x}{\varkappa} - e^{-\mu(2l-x)}\sin\frac{\sigma}{\varkappa}(2l-x)}{e^{-\mu x}\cos\frac{\sigma x}{\varkappa} - e^{-\mu(2l-x)}\cos\frac{\sigma}{\varkappa}(2l-x)}, \tag{19}$$

$$= \frac{\tan\frac{\sigma l}{\varkappa} - \tan\frac{\sigma(l-x)}{\varkappa}\coth\mu(l-x)}{1+\tan\frac{\sigma l}{\varkappa}\tan\frac{\sigma(l-x)}{\varkappa}\coth\mu(l-x)}.$$

We may, therefore, write

$$\psi = \frac{\sigma l}{\varkappa} - \vartheta,$$

$$\text{if} \quad \tan\vartheta = \tan\frac{\sigma(l-x)}{\varkappa}\coth\mu(l-x). \tag{20}$$

This gives

$$\tau_1 = \tau + \frac{\sigma l}{\varkappa} - \vartheta. \tag{21}$$

The phase-difference between maximum current and high water is:

$$\tau_1 - \tau_0 = \tau + \vartheta_0 - \vartheta.$$

As we have

$$\tan(\vartheta - \vartheta_0) = \frac{\tan\frac{\sigma}{\varkappa}(l-x)\left[\coth\mu(l-x) - \tanh\mu(l-x)\right]}{1+\tan^2\frac{\sigma}{\varkappa}(l-x)} = \frac{\sin\frac{2\sigma}{\varkappa}(l-x)}{\sinh 2\mu(l-x)}, \tag{22}$$

the phase-difference which is caused by the reflection is

$$\tan^{-1}\left(\frac{\sin\frac{2\sigma}{\varkappa}(l-x)}{\sinh 2\mu(l-x)}\right). \tag{23}$$

This expression has its greatest value when $x = l$, where the limit is

$$\tan^{-1}\left(\frac{1}{p}\right) = \frac{\pi}{2} - \gamma.$$

When there is no resistance the phase-difference is $\frac{\pi}{2}$ as it ought to be. For greater distances from the shore the phase-difference tends to zero.

To calculate the orientation of the tidal ellipses we have the formula

$$\tan 2\chi_1 = \frac{2(M_1P_1 + N_1Q_1)}{M_1^2 + N_1^2 - P_1^2 - Q_1^2}, \tag{24}$$

or as

$$M_1P_1 + N_1Q_1 = (MP + NQ)(\alpha^2 + \beta^2)$$
$$M_1^2 + N_1^2 - P_1^2 - Q_1^2 = (M^2 + N^2 - P^2 - Q^2)(\alpha^2 + \beta^2),$$

we get $\chi_1 = \chi$ as for the incident wave alone. The orientation remains, therefore, unchanged.

We may now calculate the time which the wave takes to travel the distance l. For ζ we find

$$(\tau_0)_{x=l} = \frac{\sigma l}{\varkappa}$$

$$(\tau_0)_{x=0} = \frac{\sigma l}{\varkappa} - \vartheta_0$$

and, therefore,

$$(\tau_0)_{x=l} - (\tau_0)_{x=0} = \vartheta_0 \tag{25}$$

where

$$\tan\vartheta_0 = \tan\frac{\sigma l}{\varkappa}\tanh\mu l .$$

If μl is great we have $\tanh\mu l = 1$ and $\vartheta_0 = \frac{\sigma l}{\varkappa}$ or $t = \frac{l}{\varkappa}$ as in the case of no reflection. For the maximum velocity we get

$$(\tau_1)_{x=l} = \tau + \frac{\sigma l}{\varkappa} - \frac{\pi}{2} + \gamma$$

$$(\tau_1)_{x=0} = \tau + \frac{\sigma l}{\varkappa} - \vartheta_1 \tag{26}$$

and, therefore,

$$(\tau_1)_{x=l} - (\tau_1)_{x=0} = \vartheta_1 + \gamma - \frac{\pi}{2} \tag{27}$$

$$\tan\vartheta_1 = \tan\frac{\sigma l}{\varkappa}\coth\mu l .$$

For great values of μl we may again put $\coth\mu l = 1$ and the phase-difference for $x = l$ and $x = 0$ is

$$\frac{\sigma l}{\varkappa} + \gamma - \frac{\pi}{2}$$

as $\gamma < \frac{\pi}{2}$, we find that the time interval is shorter for the velocity than for high water.

This point has some importance when one tries to draw conclusions on the course of the cotidal lines from current measurements. We see that when reflection takes place the phase-difference between maximum current and high water may have any value smaller than $\frac{\pi}{2} - \gamma$ and that the ratio between V and ζ may vary. The position of the cotidal lines can not be fixed on the basis of current measurements if we do not know the phase-difference, which is caused by the reflection.

19. Current Measurements.

Before we go to a discussion of the observations we may recall the principal results of the theory. It was found that the current vector describes an ellipse and that the rotation is in the negative direction. The ratio of the axes varies with the depth, being small near the bottom and increasing upwards when the surface is free. The time of maximum current always occurs earlier than high water. The phase-difference is greatest near the bottom and decreases upwards. The direction of

the maximum current also varies with the depth. Near the bottom this direction nearly coincides with the direction of progress of the wave. If we then go upwards it is deflected to the right, and if the depth is great it will reach a maximum deflection. At great distances from the bottom the direction of maximum current and the direction of propagation are nearly the same.

The named effects increase with the viscossity and also with $s = \frac{\lambda}{\sigma}$.

If the influence of the Earth's deflecting force be partly balanced by the slope of the wavecrest, i. e. $\theta > 0$, all the effects are diminished except the phase-difference between maximum current and high water which is independent of θ.

When comparing the observed currents with the computed, we must bear in mind that the observed values naturally must be affected with relatively great errors. The propeller-currentmeters which were used will not record currents which are weaker than about 3 cm. per sec.

We also remember that the directions are not recorded continuously, but that the Ekman-current-meter is divided in 36 segments which cover 10° each, and that the electric recording current-meter had only 18 segments, covering 20° each. On the other hand the electric current-meter had the advantage of giving a continuous record, and that the hourly means, therefore, have an accuracy of about 5°. We must expect that weak currents are affected with relatively great errors and that the directions may be erroneous to the above named degree.

Still greater errors may be introduced by the manner of computation. When the recorded current is decomposed into its N—S and E—W-components, the curves may be more or less skew, and before analysing the curves a correction after Lamont has been applied. But if this skewnes is not correctly estimated, it may have the effect of turning the current-ellipse an angle which is far greater than that which may originate from a defective record. The facts which we may ascertain from the measurements are mainly the following:

The currents change continually in direction and strength. The currents are rotatory and the direction of the rotation is negative or clockwise. The velocity decreases near the bottom. If the surface be ice-covered the velocities are greatest in some intermediate depth. The time of maximum current varies with the depth is ann generally earlier near the bottom. The direction of the maximum current also changes with the depth though less regularly, and this may to some degree be due to the mode of registration of the current.

The apparent velocity of propagation is generally greater than the value $c = \sqrt{gh}$. This velocity depends on the shape of the current ellipse, being great when the ellipse is broad and smaller when the ellipse is narrow. We shall see that in the simpler cases we may find solutions which give a similar distribution as that observed.

20. Station 8.

The current measurements which are used here for comparison of theory and observations are all provided by Sverdrup. The computation of the results are also done by him exept for station 9 which has been computed by the writer.

The surface current was determined by means of a sounding lead in the way described by Sverdrup.[1]) The relative velocities in the different depths were found by means of the electric recording current meter or an Ekman current meter.

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[1]) L. c. pag. 7.

One of the most interesting cases is his St. 8. July 18, 1924. The depth is only 22 m. The tidal currents are strong, more than 40 cm. per sec. and the current is approximately alternating. The influence of the eddy viscossity is very great as the ice-covered surface also makes a resistance to the movement. The values found by Sverdrup are:

Depth m.	Max. tidal current			Min. current	Min./Max.	Direct. of rotation
	cm/sec.	Against true	Tidal hour			
0	41.5	S 50 E	3.1	1.0	0.02	c. cl.
12	42.5	S 44 E	2.8	10.5	0.25	cl.
Mean:						
0—22.....	38.0	S 45 E	3.0	5.0	0.13	cl.

The fact that the current is stronger in 12 m. than at the surface indicates that the ice-surface offers a resistance which we must take into account if we wish to compute an example whih corresponds to the observed current. At first I have computed an example on the basis of the assumptions:

$$\nu = 390, \quad fh = 1.5, \quad s = 1 \text{ and } \theta = 0.16.$$

With these values we find

$$p = 0.7488 \qquad \vartheta = 1.529 \qquad \varkappa = 9{,}6 \text{ m/sec.}$$

For the mean velocities we obtain

$$M = 5.372, \quad N = -3.854, \quad P = -1.614, \quad Q = 0.800,$$

and for V = 38 cm. per. sec. we find C = 103.8 cm. The value found by observation is 105 cm.

The current rotates clockwise and the minimum velocity is 1.56 cm. This is considerably smaller than the observed value. To get a better agreement we might either make θ smaller or reduce the coefficient of eddy viscossity, but in either case we should get a smaller amplitude.

The time difference between maximum current and high water is calculated by means of the formulae (9. 1).

We find $\iota = -35°$ corresponding to 1 h.12 m. The observed difference is 2 h. The variation with depth is seen in the following table.

	V	αV	χ	τ
0	41.4 cm.	— 1.8	— 15°.8	— 34°.8
12	44.6 „	— 1.8	— 14°.9	— 35°.2
17	27.8 „	— 1.2	— 13°.8	— 36°.6
Mean	38 „	— 1.56	— 15°	— 35°

Another example has been calculated with the assumptions

$$\nu = 300.67, \text{ or } \beta = 0.65 \cdot 10^{-3}, \; fh = 1.5, \; s = 1 \text{ and } \theta = 0.$$

We then find $p = 0.7356$, $\vartheta = 1.448$, $\varkappa = 10.2$ m/sec.

If we assume that the mean velocity is 38 cm. per sec. we get C = 96. The variation with depth is seen from the following table:

	V	σV	χ	τ
0	41.7	— 6.2	— 21°.2	— 32°.5
12	44.6	— 6.1	— 20°.0	— 33°.7
Mean	38.0	— 5.2	— 20°.2	— 33°.5

We see that there are still some discrepancies between the observed and the calculated values. To judge the agreement we must remember that the distance from the shore is short, and it is, therefore, reasonable to think that the ice-surface does not make the same resistance to movement in all directions, but may move comparatively freely parallel to the shore, but not in the direction perpendicular to the shore.

The surface conditions, therefore, ought to have been

$$\frac{\partial u}{\partial z} + fu = 0$$

$$\frac{\partial v}{\partial z} + f_1 v = 0 \qquad \text{where } f_1 > f.$$

This condition would make the computations more difficult, and it is, therefore, not attempted.

The discrepancy between the computed and the observed tidal hours can not be accounted for in this way. This discrepancy is to some degree explained by Sverdrup. But we might also take a reflected wave into consideration. If the reflection took place perpendicular to the shore at a distance l from the place, the ratio $\frac{V}{C}$ must be multiplied with $\sqrt{\frac{\cosh 2\mu l - \cos\frac{2\sigma l}{\varkappa}}{\cosh 2\mu l + \cos\frac{2\sigma l}{\varkappa}}}$ and the phase-difference should be augmented by

$$\tan^{-1}\left(\frac{\sin\frac{2\sigma l}{\varkappa}}{\sinh 2\mu l}\right).$$

And we may therefore easily understand that a suitable reflection might account for the observed amplitude and tidal hours, even if we reduce the coefficient of eddy viscossity.

21. Station 9.

This station is situated at 76° 36′ N Lat. and 138° 30′ E Long. The depth is 22 m. The current measurements were made on August 1. 1924. The surface current was determined by means of a sounding lead and the relative velocities in 12 m. and 20 m. were found by means of an Ekman current-meter. The material has been analysed by the writer. The epoche is 9 h. M.L.T.

If we place the x-axis towards south and the y-axis towards east, we find for the surface:

$$M = -2.6, \quad N = 22, \quad P = 19.4, \quad Q = 13.4.$$

The axes and the orientation of the ellipse were computed by means of the formulae (9, 1), which give:

$$V = 27.8 \text{ cm. per sec.} \qquad \alpha V = -16.7 \text{ cm. per sec.} \qquad \alpha = -0.6$$
$$\tau = 62°.3 \qquad \chi = 48°.8.$$

In 12 m. we find:

$$M = -4.3 \qquad N = 17.7$$
$$P = 17.1 \qquad Q = 14.5$$
$$V = 24.9 \qquad \alpha V = -14.7 \qquad \alpha = 0.59$$
$$\text{and} \quad \tau = 60°.9 \qquad \chi = 56°.3.$$

We see that the phase-difference between the current at the surface and at 12 m. is 62°.3 — 60°.9 = 1°.4, which means that the maximum velocity in 12 m. occurs 3 min. earlier than at the surface. The current ellipse at the surface is furthermore deflected 7°.5 to the right in relation to the current ellipse in 12 m.

The measurements in 20 m. show great irregularities and it is not wise to try to find the orientation and phase of the current, but they are sufficient to ascertain that the maximal velocity is some 12 cm. per sec.

In order to see how far these features may be found from the theory, I have calculated an example, and computed the velocities, at first under the assumption that the velocities are zero at the bottom, and then under the assumption that there is some slip at the bottom.

I have made the following assumptions:

$$\nu = 60 \qquad \lambda = 1.4584 \sin 76°.36' \qquad s = 1.0096$$
$$\sigma = 1.4052 \qquad s\vartheta = 0.58,$$

we then find

$$a_1 = b_1 = 0.10478 \qquad a_2 = 0.99373 \qquad b_2 = -0.072044$$
$$p = 0.352 \qquad \vartheta = 0.6129 \qquad \varkappa = 24 \text{ m/sec.}$$

The variation of the velocities is seen from the following tables:

	M	N	P	Q
0	2.9854	— 0.4779	— 0.7055	— 1 8028
12	2.3257	— 0.3524	— 0.4596	— 1.0427
20	0.6765	— 0.2249	+ 0.0323	— 0.1579

	V	αV	α	χ	τ	τ (time)
0	27.8	— 16.9	— 0.6076	— 12°.6	— 1°.3	2.6 min.
12	21.5	— 9.9	— 0.4582	— 9°.4	— 4°.2	8.4 „
20	6.5	— 1.3	— 0.1934	+ 6°.5	— 19°.5	39.0 „

The phase-difference between the velocity at the surface and at 12 m. is 2°.9 and the deflection is 3°.2, these quantities were observed to be 1°.3 and 7°.5 respectively.

The velocities in 20 meter are too small, so that the agreement is not good. To get a better agreement between observed and calculated quantities we might either

reduce the assumed coefficient of eddy viscossity or assume slipping motion at the bottom. I have calculated an example with the last assumption.

The coefficient of eddy viscossity is as in the first example $\nu = 60$, and furthermore I have assumed $fh = 4$, $s\theta = 0.57$.

We then obtain:

$$a_1 = 0.016812 \qquad b_1 = 0.05693$$
$$a_2 = 0.983123 \qquad b_2 = -0.12475$$
$$p = 0.232 \qquad \vartheta = 0.5725 \qquad \varkappa = 25.7 \text{ m/sec.}$$

For the velocities we get the following relative values:

	M	N	P	Q
0	2.7789	— 0.2813	— 0.4197	— 1.6469
12	2.3698	— 0.2276	— 0.3180	— 1.1776
20	1.4869	— 0.2142	— 0.0784	— 0.5043

	V	αV	$-\alpha$	χ	τ	τ (time)
0	27.8	— 16.5	0.594	— 8°	— 1°.1	2.2 min.
12	23.7	— 11.9	0.5	— 6°.5	— 2°.2	4.4 „
20	14.9	— 5.0	0.34	0°	— 8°.2	16.4 „

Now the phase-difference is 1°.1 between surface and 12 m. and the deflection is 1°.5. The velocity in 20 m. is 14.9 cm/sec. against the observed 12 cm. The agreement between computed and observed values is better, except the deflection which is smaller than observed. But we have to remember that the deflection is very sensitive to any adjustment of the observed values on account of the skewness of the current curves.

It seems as the hypothesis of slipping is most in accordance with the observations, but we might also have reduced the coefficient of eddy viscossity, and then got greater velocities at 20 m., so it seems uncertain which of the two hypotheses best answers to the real conditions.

22. Stations 5 and 6.

Station 5 is situated at 75° 12′ N Lat. and 159° 40′ E Long. The current measurements were executed from Nov. 24. to 27. 1923. Station 6 is situated at 75° 11′ N Lat. and 157° 40′ E Long. The measurements comprices 3 days from Febr. 6. to 9. 1924. The depths are at both stations 38 m. and the currents are determined in 28 m. The results are almost equal, the maximum current being 9.5 cm. and the minimum current at station 5 7.5 cm. and at station 6 6.5 cm. per sec.

That the minimum current should be weaker at station 6 was to be expected as it lies nearer to the New Siberian Islands. The direction of the maximum current is at both stations the same.

As the latitude is the same, we may treat both stations at the same time. As s is sensibly equal to one, we may use the somewhat simpler equations corresponding to

this case. For station 5 the ratio between the axes of the ellipse is 0.78. When trying to find a solution which corresponds to this, the alteration of two quantities are at our disposal: the indetermined quantity θ and the coefficient of eddy viscossity ν. I have computed a solution with the assumptions $\theta = 0.28$ and $\beta = \sqrt{\frac{\sigma}{\nu}} = 3.10^{-3}$ which corresponds to $\nu = 15.6$.

If we furthermore assume that the ice-cover does not take part in the motion, we find

$$\beta h = 11.4$$

$$a_1 = b_1 = \frac{1}{11.4}, \qquad b_2 = \frac{1}{6}(\beta h)^2 = 21.66,$$

$$k_1 = 1.925 \cdot 10^{-3}, \qquad k_2 = 45.9 \cdot 10^{-3},$$

$$p = 0.636, \qquad \vartheta = 0.38,$$

$$A = 0.33, \qquad B = 0.05, \qquad A_1 = B_1 = 0.1207.$$

The velocities for $z = 10$ may now be calculated. We find:

$$1 - \varphi_1 = 1.049, \quad \psi_1 = 0.007, \quad \psi_2 = 25.2$$

$$M = 3.40, \quad N = -1.14, \quad P = -1.39, \quad Q = -2.70,$$

$$V = \sqrt{\frac{g}{h}}\,\frac{C}{2}\,3.67, \quad \alpha V = -\sqrt{\frac{g}{h}}\,\frac{C}{2}\,2.93, \quad \alpha = -0.8,$$

$$\chi = -21°.1, \quad \tau = -1°.4.$$

If we take V to be 9.5 cm. as observed, we find $C = 10.2$ cm. and $\alpha V = -7.6$ cm. If we had assumed a free surface, C would have been twice as great.

For station 6 we use the same value of β but put $\theta = 0.5$. We then obtain:

$$p = 0.323, \ \vartheta = 0.5335, \ A = 0.517, \ B = 0.017, \ A_1 = B_1 = 0.086,$$

and for the velocities at $z = 10$ m. we find:

$$M = 2.71, \quad N = -0.34, \quad P = -0.52, \quad Q = -1.63,$$

$$V = \sqrt{\frac{g}{h}}\,\frac{C}{2}\,2.76, \qquad \alpha V = -\sqrt{\frac{g}{h}}\,\frac{C}{2}\,1.66, \quad \alpha = -0.6,$$

$$\chi = -10°.2, \quad \tau = -1°,$$

or if we assume $V = 9.5$ cm. we get $C = 13.6$ cm. and $\alpha V = -5.7$ cm.

The value of ν which has been used is perhaps too small, but if we assume that ν is considerably greater, it is impossible to find a solution with the ratio $-\alpha = 0.78$ as that observed at station 5. At any rate trials have shown that ν can not surpass 50.

The mean kinetic energy may be found by means of the formula

$$E_k = \frac{1}{4}\, g\varrho C^2 \left((A^2 + A_1{}^2)\,\frac{1 - a_1}{2} + (B^2 + B_1{}^2)\,\frac{1 - a_2}{2} \right).$$

In the case $s = 1$ we have to replace $1 - a_2$ by $\frac{1}{30}(\beta h)^4$ and the formula then will be

$$E_k = \frac{1}{4}\, g\varrho C^2 \left((A^2 + A_1{}^2)\,\frac{1 - a_1}{2} + (B^2 + B_1{}^2)\,\frac{1}{60}(\beta h)^4 \right).$$

For the mean kinetic energy we have the formula

$$E_p = \frac{1}{4} g \varrho C^2 .$$

For station 5 we then obtain

$$E_k = \frac{1}{4} g \varrho C^2 \cdot 4.882 \quad \text{and} \quad E_T = E_k + E_p = \frac{1}{4} g \varrho C^2 \cdot 5.882 ,$$

for C we have found 10.2 cm. If we insert $g = 982$, $C = 10.2$, we get the energy of a water-column of 1 cm^2. base

$$E_T = 150.24\, \varrho \cdot 10^3 \text{ ergs.}$$

And if we take 1 m^2. as a base, we find

$$E_T = 150.24\, \varrho \text{ joule per m}^2.$$

For station 6 we find

$$E_k = \frac{1}{4} g \varrho C^2 \cdot 2.287$$

and

$$E_T = \frac{1}{4} g \varrho C^2 \cdot 3.287 = \frac{1}{4} g \varrho \cdot 13.2^2 \cdot 3.287 = 149.33\, \varrho \cdot 10^3 \text{ ergs.}$$

The energy is therefore sensibly equal at both stations.

The tidal range is not observed at these stations, but we may compare the values here found with the values observed at st. 3. By means of Simpson's formula and interpolated values of the velocities I have found that the mean kinetic energy at st. 3 is $81.29 \cdot 10^3 \varrho$ ergs and as the amplitude of the surface-elevation is 9 cm., we find that the mean potential energy is $19.89 \cdot 10^3$ ergs. The total energy is, therefore, $101.18 \cdot 10^3$ ergs per cm^2. Now the distance between st. 3 and st. 5 is about 190 km. along the cotidal line, and the time-difference is 0.9^h. Along the cotidal line the energy increases and may be computed by means of the formula:

$$E_1 = E e^{2my}$$

here $m = \frac{\lambda \theta}{c}$ which is about $1.9 \cdot 10^{-6}$ per m. We therefore have $E_1 = E e^{0.722}$ and in 0.9^h this value is reduced by damping according to the formula

$$E_2 = E_1 e^{-2\mu x} = E_1 e^{-2\frac{2\pi}{12.4} 0.9\, p} .$$

We thus get

$$E_2 = E e^{0.722 - 0.58} = 1.1526\, E = 116.62 \cdot 10^3 \text{ ergs.}$$

This corresponds to an amplitude $C = 9$ cm. at st. 5. The value is in fair agreement with the computed amplitude $C = 10.2$ cm.

23. Station 3.

We have seen that the configuration of the motion will be modified to some extent, if the eddy viscossity varies with the depth. This is especially the case if the surface is ice-covered. Such a case has been observed by Sverdrup at his St. 3[1]).

[1]) L. c. pag. 9.

Depth m.	Max. tidal current			Min. tidal current m/sec.	Ratio: Min./Max.	Direct. of rotat.	Number of half day periods	Remarks
	m/sec.	Against (true)	Tidal hour					
0—30								Too weak to be observed.
35	6.2	S 50° W	8.9	5.4	0.87	cl.	4	
42	15.4	S 73° W	8.9	12.9	0.84	cl.	4	
46	11.0	S 29° W	7.2	8.8	0.80	cl.	2	
50	5.4	N 87° W	6.8	2.8	0.52	cl.	2	

He has also been able to explain the most peculiar features as caused by the distribution of the density. To get the approximate velocities he has considered the sea as made up of three layers of different viscossity. Here we shall see that we may get a similar distribution of the velocities if we assume that the coefficient of eddy viscossity varies continously with the depth such that the eddy viscossity is zero at some depth.

We take the eddy viscossity in the bottom layer to be expressed by

$$\nu = \nu_1 \left(1 - \frac{\alpha z}{h}\right)^{\frac{2n+1}{n+1}}$$

and in the top layer by

$$\nu = \nu_2 \left(\frac{\frac{\alpha z}{h} - 1}{\alpha - 1}\right)^{\frac{2n+1}{n+1}} .$$

To facilitate the computations we asume

$$\nu_2 = \nu_1 (\alpha - 1)^2 .$$

In the bottom-layer we put

$$x = \left(1 - \frac{\alpha z}{h}\right)^{\frac{1}{2(n+1)}}, \quad z = \frac{h}{\alpha}\left(1 - x^{2(n+1)}\right)$$

and in the top-layer

$$x = \left(\frac{\frac{\alpha z}{h} - 1}{\alpha - 1}\right)^{\frac{1}{2(n+1)}} \qquad z = \frac{h}{\alpha}\left(1 + (\alpha - 1)\, x^{2(n+1)}\right) .$$

We then get the equation

$$x^{-(2n+1)} \frac{d}{dx}\left(x^{2n+1} \frac{d\Phi}{dx}\right) - 2i\left(2(n+1)\frac{\beta h}{\alpha}\right)^2 \Phi = 0, \tag{1}$$

or

$$\frac{d^2\Phi}{dx^2} + \frac{2n+1}{x}\frac{d\Phi}{dx} - 2i\varrho^2\Phi = 0, \qquad \varrho = 2(n+1)\frac{\beta h}{\alpha} .$$

The boundary conditions take the form:

$$\Phi = 1, \quad \text{for } x = 1 \quad \text{and} \quad \Phi \text{ finite for } x = 0.$$

The last condition may be replaced by

$$\nu \frac{d\Phi}{dz} = 0, \qquad \text{for } z = \frac{h}{\alpha}.$$

The integral is

$$x^{-n} \frac{J_n(\varrho x(1-i))}{J_n(\varrho(1-i))}. \tag{2}$$

Where J_n is Bessel's function of n^{th} order. Furthermore we assume $\sigma = \lambda$ or $s = 1$. $1 - \Phi_2$ must then be replaced by $i\psi_2$ where ψ_2 is determined by

$$\frac{d}{dz}\left\{\left(1 - \frac{\alpha z}{h}\right)^{\frac{2n+1}{n+1}} \frac{d\psi_2}{dz}\right\} = -2\beta^2. \tag{3}$$

Integrating we get

$$\left(1 - \frac{\alpha z}{h}\right)^{\frac{2n+1}{n+1}} \frac{d\psi_2}{dz} = -2\beta^2(z + c),$$

and $\nu \frac{d\psi}{dz} = 0$ for $z = \frac{h}{\alpha}$ gives $c = -\frac{h}{\alpha}$.

We then obtain

$$\frac{d\psi_2}{dz} = \frac{2\beta^2 h}{\alpha}\left(1 - \frac{\alpha z}{h}\right)^{-\frac{n}{n+1}}.$$

Integrating again we find

$$\psi_2 = 2\frac{\beta^2 h^2}{\alpha^2}(n+1)\left(1 - \left(1 - \frac{\alpha z}{h}\right)^{\frac{1}{n+1}}\right)$$

or

$$\psi_2 = 2\frac{\beta^2 h^2}{\alpha^2}(n+1)(1 - x^2) \tag{4}$$

where $$x = \left(1 - \frac{\alpha z}{h}\right)^{\frac{1}{2(n+1)}} \quad \text{for} \quad z < \frac{h}{\alpha}$$

and $$x = \left(\frac{\frac{\alpha z}{h} - 1}{\alpha - 1}\right)^{\frac{1}{2(n+1)}} \quad \text{for} \quad z > \frac{h}{\alpha}.$$

For A_1 we get

$$A_1 = \frac{1}{h}\int_0^h \Phi_1\, dz = \frac{1}{h}\int_0^{\frac{h}{\alpha}} \Phi_1\, dz + \frac{1}{h}\int_{\frac{h}{\alpha}}^h \Phi_1\, dz.$$

In the interval $0 < z < \frac{h}{\alpha}$ we have

$$dz = -\frac{2(n+1)\,h}{\alpha}\,x^{2n+1}\,dx$$

and in the interval $\frac{h}{\alpha} < z < h$ we have

$$dz = \frac{2\,(n+1)\,h}{\alpha}\,(\alpha - 1)\,x^{2n+1}\,dx.$$

We therefore get

$$A_1 = \frac{2\,(n+1)}{\alpha}\int_0^1 \Phi_1\,(x)\,x^{2n+1}\,dx + \frac{2(n+1)}{\alpha}(\alpha - 1)\int_0^1 \Phi_1\,(x)\,x^{2n+1}\,dx$$

or as

$$x^{2n+1}\,\Phi_1\,(x) = \frac{1}{2i\varrho^2}\,\frac{d}{dx}\left(x^{2n+1}\,\frac{d\Phi_1}{dx}\right),$$

we obtain

$$A_1 = \frac{2\,(n+1)}{2\,i\varrho^2}\int_0^1 \frac{d}{dx}\left(x^{2n+1}\,\frac{d\Phi_1}{dx}\right)dx = \frac{2\,(n+1)}{2\,i\varrho^2}\left(\frac{d\,\Phi_1}{dx}\right)_{x=1}. \qquad (5)$$

A well known formula for the Bessel's functions gives

$$\frac{d\Phi}{dx} = \frac{d}{dx}\left(x^{-n}\,\frac{J_n\,(\varrho x\,(1-i))}{J_n\,(\varrho\,(1-i))}\right) = -\,\varrho\,(1-i)\,\frac{J_{n+1}\,(\varrho x\,(1-i))}{J_n\,(\varrho\,(1-i))}$$

or

$$A_1 = \frac{i\,(1-i)\,2\,(n+1)}{2\cdot 2\,(n+1)\,\frac{\beta h}{\alpha}}\,\frac{J_{n+1}\,(\varrho\,(1-i))}{J_n\,(\varrho\,(1-i))} = \frac{(1+i)\,\alpha}{2\,\beta h}\,\frac{J_{n+1}\,(\varrho\,(1-i))}{J_n\,(\varrho\,(1-i))}. \qquad (6)$$

For b_2 we get

$$b_2 = \frac{1}{h}\int_0^{\frac{h}{\alpha}} \psi_2 dz + \frac{1}{h}\int_{\frac{h}{\alpha}}^{h} \psi_2 dz = 2(n+1)\int_0^1 x^{2n+1}\,\psi_2(x)\,dx$$

$$= 4(n+1)^2\,\frac{\beta^2 h^2}{\alpha^2}\int_0^1 (x^{2n+1} - x^{2n+3})\,dx = 2\,\frac{\beta^2 h^2}{\alpha^2}\,\frac{n+1}{n+2}. \qquad (7)$$

For $x = 1$ the argument of the Bessel's functions will generally be great and the series will converge very slowly, we may, therefore, use the asymptotic expansions for J. We have

$$2J_n = H_n^{(1)}\,(x) + H_n^{(2)}\,(x)$$

and

$$H_n^{(1)}(re^{i\vartheta}) = \frac{e^{-r\sin\vartheta + i\left(r\cos\vartheta - \frac{2n+1}{4}\pi - \frac{\vartheta}{2}\right)}}{\sqrt{\frac{1}{2}\pi r}} S_n\left(re^{i\left(\vartheta - \frac{\pi}{2}\right)}\right)$$

$$H_n^{(2)}(re^{i\vartheta}) = \frac{e^{r\sin\vartheta - i\left(r\cos\vartheta - \frac{2n+1}{4}\pi + \frac{\vartheta}{2}\right)}}{\sqrt{\frac{1}{2}\pi r}} S_n\left(re^{i\left(\vartheta + \frac{\pi}{2}\right)}\right) \tag{8}$$

$$S_n(x) = 1 + \sum \frac{(4n^2-1)(4n^2-9).....(4n^2-(2m-1)^2)}{m!\,(8x)^m}. \tag{9}$$

Here we have

$$re^{i\vartheta} = \varrho x(1-i) \quad \text{or} \quad r = \varrho x\sqrt{2}, \quad \vartheta = -\frac{\pi}{4}.$$

We therefore get

$$-r\sin\vartheta + ir\cos\vartheta = \varrho x(1+i)$$

$$re^{i\left(\vartheta+\frac{\pi}{2}\right)} = \varrho x(1+i), \quad re^{i\left(\vartheta-\frac{\pi}{2}\right)} = -\varrho x(1+i).$$

To abridge the writing we put

$$\varrho x(1+i) = y, \quad \varrho(1+i) = y_1, \quad \left(\frac{2n+1}{4} - \frac{1}{8}\right)\pi = \omega_1, \quad \left(\frac{2n+1}{4} + \frac{1}{8}\right)\pi = \omega_2.$$

We then get

$$\Phi = x^{-\frac{2n+1}{2}} \frac{e^{y-i\omega_1} S_n(-y) + e^{-y+i\omega_2} S_n(y)}{e^{y_1-i\omega_1} S_n(-y_1) + e^{-y_1+i\omega_2} S_n(y_1)}.$$

or as $\omega_1 + \omega_2 = \frac{2n+1}{2}\pi$

$$e^{i(\omega_1+\omega_2)} = e^{n\pi i + \frac{i\pi}{2}} = (-1)^n i$$

$$\Phi_1 = x^{-\frac{2n+1}{2}} \frac{e^{-(y_1-y)} S_n(-y) + (-1)^n ie^{-(y_1+y)}}{S_n(-y_1) + (-1)^n ie^{-2y_1} S_n(y_1)} S_n(y)$$

$$= x^{-\frac{2n+1}{2}} e^{-(y_1-y)} \frac{S_n(-y) + (-1)^n ie^{-2y} S_n(y)}{S_n(-y_1) + (-1)^n ie^{-2y_1} S_n(y_1)}.$$

This may often be replaced by the approximate value

$$\Phi_1 = x^{-\frac{2n+1}{2}} e^{-(y_1-y)} \frac{S_n(-y)}{S_n(-y_1)}.$$

or if we again write $y = \varrho x(1+i)$, $y_1 = \varrho(1+i)$

$$\Phi_1 = x^{-\frac{2n+1}{2}} e^{-\varrho(1+i)(1-x)} \frac{S_n(-\varrho x(1+i))}{S_n(-\varrho(1+i))}.$$

For small values of the argument we must use the development in convergent series

$$J_n(\varrho x(1-i)) = \left(\frac{\varrho x(1-i)}{2}\right)^n \sum (2i)^m \frac{\left(\frac{\varrho x}{2}\right)^{2m}}{m!(m+n)!}. \tag{12}$$

The denominator may then be calculated by the formula

$$J_n(\varrho(1-i)) \sim \frac{e^{\varrho(1+i) - i\left(\frac{2n+1}{4}\pi - \frac{\pi}{8}\right)}}{\sqrt{2\pi\varrho\sqrt{2}}}. \tag{13}$$

The asymtotic value of A_1 is

$$A_1 = \frac{(1-i)\alpha}{2\beta h} \frac{S_{n+1}(-\varrho(1+i))}{S_n(-\varrho(1+i))}. \tag{14}$$

We shall use these formulae to compute a numerical example. We assume

$$\beta = 10^{-3} \quad h = 5.6 \cdot 10^3 \text{ cm.} \quad n = 2 \quad \theta = 0,$$

which gives

$$\varrho = 8.4.$$

With these values we find

$$S_2(-\varrho(1+i)) = 0.88825 + i\,0.10567,$$
$$S_3(-\varrho(1+i)) = 0.74108 + i\,0.20953,$$
$$S_2\left(-\varrho\frac{1}{2}\sqrt{3}(1+i)\right) = 0.87092 + i\,0.12092,$$
$$A_1 = a_1 - ib_1 = 0.367 - i\,0.1279, \quad b_2 = 2.94,$$
$$p = 0.8147, \quad \vartheta = 0.876, \quad \varkappa = 26.77 \text{ m. per sec.}$$

To see the variation of the velocities with depth the following tables have been computed.

d	Depth z	φ_1	ψ_1	ψ_2	M	N	P	Q
0	56	1	0	0	0	0	0	0
24.3	31.7	0.08424	0.1857	2.94	3.46	— 2.28	— 3.60	— 2.33
36.75	19.25	— 0.1337	0.09311	5.88	6.00	— 5.05	— 6.71	— 3.58
42	14	0.0369	0.01082	11.76	10.55	— 10.99	— 12.53	— 8.61
43.75	12.25	— 0.1397	0.09311	5.88	6.00	— 5.05	— 6.71	— 3.58
47.9	8.1	0.08424	0.1857	2.94	3.46	— 2.28	— 3.60	— 2.33
56	0	1	0	0	0	0	0	0
Mean		0.367	0.1279	2.94	3.13	— 2.55	— 3.33	— 1.66

Depth	V	αV	α	χ	τ	τ (time)
0	0	0	—	— 9°.6	— 48°.2	1 h. 40 m.
24.3	5.04	— 2.54	— 0.503	— 39°.3	— 11°.5	24 „
36.75	9.05	— 6.12	— 0.676	— 42°.7	— 8°.1	16.8 „
42	16.41	— 13.93	— 0.849	— 44°.7	— 6°.1	12.6 „
43.75	9.05	— 6.12	— 0.676	— 42°.7	— 8°.1	16.8 „
47.9	5.04	— 2.54	— 0.503	— 39°.3	— 11°.5	24 „
56	0	— 0	—	— 9°.6	— 48°.2	1 „ 40 „
Mean	4.63	— 2.96	— 0.64	— 39°.2	— 11°.16	24 „

We see that the velocities have a sharp maximum at 42 m. and decreases rapidly especially towards the bottom. The deflection of the orbital ellipse also reaches a maximum at 42 m. where it is about 45°. At the surface and at the bottom this deflection tends to limit $\chi = -9°.6$.

The phase-difference has a minimum at 42 m. and attains its greatest value at the bottom and at the surface where the time-difference between high water and maximum current tends to the limit 1 h. 40 m.

Under extreme conditions the distribution of velocity may be very peculiar.

Suppose that the eddy viscossity is zero at some intermediate depth. We make the following assumption

$$\nu = \nu_0 \left(1 - \frac{\alpha z}{h}\right)^2, \qquad \alpha > 1.$$

We then have

$$\Phi = \left(1 - \frac{\alpha z}{h}\right)^r, \qquad z < \frac{h}{\alpha},$$

where r is the root of the equation:

$$r(r+1) = 2i\frac{\beta^2 h^2}{\alpha^2}, \quad \beta = \sqrt{\frac{\sigma + \lambda}{2\nu_0}},$$

whose real part is positive. For $z > \frac{h}{\alpha}$ we get:

$$\Phi = \left(\frac{\frac{\alpha z}{h} - 1}{\alpha - 1}\right)^r$$

Φ satisfies the conditions:

$$\Phi = 1, \quad z = 0, \quad \text{and } z = h, \quad \Phi \text{ finite for } z = \frac{h}{\alpha}.$$

In this case we cannot assume $s = 1$, as the solution will not then remain finite for $z = \frac{h}{\alpha}$.

If we put

$$x = 1 - \frac{\alpha z}{h}, \qquad z < \frac{h}{\alpha},$$

$$x = \frac{\frac{\alpha z}{h} - 1}{\alpha - 1}, \qquad z > \frac{h}{\alpha},$$

$$r = r' + ir'',$$

we have

$$\Phi_1 = x^{r_1' + ir_1''} = \varphi - i\psi,$$

or

$$\varphi = e^{-r' \ln \frac{1}{x}} \cos\left(r'' \ln \frac{1}{x}\right), \qquad \psi = e^{-r' \ln \frac{1}{x}} \sin\left(r'' \ln \frac{1}{x}\right).$$

As a numerical example I have chosen

$$s = 0.96 \quad h = 50 \text{ m.} \quad \alpha = 5 \quad \nu_0 = 15.6 \quad \theta^2 = 0.19.$$

We then find

$$r_1 = 1.6299 + i\,2.074$$
$$r_2 = 0.02823 + i\,0.17038$$
$$p = 0.4637 \quad \vartheta = 0.6733 \quad \varkappa = 32.91 \text{ m/sec.} \quad m = \frac{\lambda\theta}{c}.$$

With these values the following table has been computed

z	0	5	9	10	11	14	30	50
V	0	4.0	9.9	24.5	15.1	9.9	4.0	0
$-\alpha$V	0	1.5	7.5	22.0	12.6	7.5	1.5	0
$-\alpha$		0.373	0.75	0.898	0.835	0.75	0.373	
τ	$-32°.8$	$-14°.5$	$-0°.5$	$+33°.4$	$+1°.9$	$-0°.5$	$-14°.5$	$-32°.8$
χ	$+3°.0$	$-11°.9$	$-17°.8$	$+17°.4$	$-14.°$	$-17°.8$	$-11°.9$	$+3.°0$

We note that the velocities have a sharp maximum at 10 m. But the most interesting detail is the orientation.

The major axis of the current ellipse close at the bottom is deflected 3° to the left. At 5 m. the deflection is 11°.9 to the right. At 9 m. — 17°.8, but at 10 m. the deflection is 17°.4 to the left. From 10 m. to 50 m. the same deflections occur in the inverse order.

The phase-difference also shows extreme variations. Near the bottom maximum current occurs an hour before high water, but at 10 m. the maximum current occurs one hour after high water.

In reality we know that the eddy visossity ν is always different from zero, so that such extreme conditions are never met with in nature. If the density changes very rapidly we may nevertheless expect that the eddy viscossity is reduced to values which correspond to the molecular value $\eta = 0.0178$, which may be 1000 or 10 000 times smaller than the eddy viscossity in homogenous water.

This example was computed with the intention of explaining some of the characteristic features of St. 3. Thus we have the observed tidal hours for 42 m. 8.9 h., for 46 m. 7.2 h. and for 50 m. 6.8 h. If in the example we take the tidal hour for $z = 10$

to be 8.9 h., we find that the tidal hour for $z = 5$ is 7.2 h. and close at the bottom 6.6 h. The tidal hour of high water is 7.8. The maximum current for $z = 10$ occurs 1.1 h. after high water. By hourly soundings the tidal hour of high water is found to be 8.2. Thus the observed current for 42 m. also occurs after high water. The resemblance is quite striking.

For the orientation we have the observed values in 42 m. S 73 W, in 46 m. S 29 W and in 50 m. N 87 W. The current in 46 m. is thus deflected very much to the left in comparison to the current in 42 m. and in 50 m. In the example we see that the current for $z = 10$ is deflected 35° to the left in comparison to the current for $z = 14$ and $z = 9$.

Too much stress should not be put on this accordance as the determination of the orientation is more difficult than the determination of the tidal hour.

At his station 4 Sverdrup also found an irregular distribution of the velocities.

But as the measurements show that the surface current varies irregularly from one day to another, it is uncertain if we may conclude that the mean values of his measurements represent a permanent phenomenon.

The computation of his measurements gives the following results: At the surface the maximum velocity is 8.1 cm. per sec. and the minimum velocity 5.8 cm/sec. The direction of maximum velocity is S 6° W and the time of maximum velocity 7 h. 38 m.

At 42 m. depth it was found:

Maximum velocity 11.3 cm., minimum velocity 8.4 cm. and direction S 45.4 W. Time of maximum velocity 6 h. 45 m.

At 42 m. the velocity attains its maximum earlier than at the surface and the difference is 53 min., and the current is deflected to the right as much as 39°.4 compared with the surface current.

This is certainly in qualitative agreement with the theory, but the amounts are much greater than one should expect. The fact that the current is stronger at 42 m. than at the surface, indicates that there is a resistance at the surface, and also that the eddy viscossity is greater near the surface than at 42 m.

In order to see if the theory may give a similar distribution of the velocities I have calculated an example.

It was assumed that the eddy viscossity increased from the bottom and upwards according to an exponential law. Perhaps it would have been better to assume that the eddy viscossity decreased from the bottom to 42 m. and then again increased to the surface. But this supposition makes the integration difficult. The conditions are then as follows:

$$\nu = \nu_0 \, e^{\frac{\alpha z}{h}}$$

$$u = v = 0 \quad \text{for} \quad z = 0,$$

$$\frac{\partial u}{\partial z} + fu = 0 \qquad \frac{\partial v}{\partial z} + fv = 0 \qquad \text{for } z = h .$$

The formula for Φ is rather complicated, but if we take the asymptotic expressions, it will be considerably simplified. For simplicity we have assumed $s = 1$.

With $\nu_0 = 50$ we then get at the surface $\tau = -8°.6$ $\chi = -11°.8$.
and at 42 m. $\tau = -2°.6$ $\chi = -24°$.

The maximum velocity at the surface occurs 12 min. earlier than at 42 m. This is in contradiction to the observed values. The current at 42 m. is deflected 6° to the right, this is qualitatively in agreement with the observations, but the amount is too small.

We see that the theory is unable to explain the surface current, but in this case we know that the surface current changed irregularly from one day to another, and the reason is probably that the ice-resistance has a very complex character which cannot be expressed by simple formulae as those above.

24. Some Remarks Regarding the Eddy Viscossity and the Dissipation.

The eddy viscossity is a quantity which is still comparatively little known, but as it is of great importance for all oceanic motion, every method which gives us some information of this quantity, is of interest. In the theory of tidal motion we have found that the eddy viscossity has an influence upon the velocity of propagation, upon the damping of the wave, and also upon the phase of the current and the orientation of the current-ellipses. A detailed investigation of these quantities may thus give us some information as to the value of the eddy viscossity. But, on the other hand, the theory has shown that the determination of the eddy viscossity is difficult as it may be a function of the depth. The eddy viscossity must depend on the stability of the water-layers, or if ϱ denotes the density, ν must be some function of $\frac{d\varrho}{dz}$, such that great numerical values of $\frac{d\varrho}{dz}$ correspond to small values of ν and vice versa.

The values of ν which have been used in the preceeding examples lie between 15 and 400. The value 15 which was used in the example which illustrates the motion at station 5, is probably too small, but corresponds to a distribution of density which shows great stability. The value 390 which was used at st. 8 was found too great. However, the values seem to be of the correct order of magnitude.

The dissipation of energy is proportional to ν and is furthermore proportional to the square of the amplitude. Thus we find that the dissipation of energy at st. 5 is 2.7 ergs per square cm. per second. At st. 9 we find 11 ergs and at st. 8 597 ergs. It must be admitted that the last value is probably too great.

A direct calculation for st. 3 by means of the formula:

$$2F = \int_0^h \nu \left[\left(\frac{dM}{dz} \right)^2 + \dots \right] dz$$

gives $0.34\,\varrho\,\nu$ ergs.

A value of 100 for $\nu\varrho$ gives 34 ergs per square cm. This value may be too large, but at any rate the energy-quantum which is dissipated is by no means insignificant, especially as the greater part of the dissipation takes place in the bottom-layer, and the heat which is produced, is not easily transmitted to the upper layers. If the energy 34 ergs per sq. cm. is used to heat a layer of 5 m. thicknes and no heat is lost by conduction, the temperature would rise 0°.05 C in one year. Actually the bottom-layer on the Siberian shelf has a temperature which lies slightly above the freezing-temperature, which the upper layers show almost uniformly, but if this can be attributed to the dissipation seems uncertain.

25. Summary of Results.

A theory of free tidal waves is set forth, taking the influence of the Earth's rotation and the eddy viscossity into account. Neither the exact theory of waves in canals, nor of waves on an unlimited rotating sheet of water, is able to explain all the observed phenomena. The solution is therefore given in an indeterminate form, which makes it possible to adjust it to the observed phenomena.

It is found that the current is rotatory, and the direction of rotation is negative (clockwise on the northern hemisphere). The ratio of the semi-axes of the current-ellipse, α, is variable with the depth, such that α has its smallest value at the bottom and increases with the distance from the bottom.

The current is characterized by the semi-axes of the ellipse V and αV, the angle χ which the major axis forms with the direction of propagation and the phase-difference τ between current and surface-elevation. It is found that the ellipse is deflected to the right, and that this deflection increases from the bottom and upwards, but may attain a maximum and then decrease again.

The current reaches its maximum before high water and the phase-difference is greatest at the bottom.

If the surface be ice-covered, it will play the same role as the bottom. The velocities are then greatest at some intermediate depth.

When the eddy viscossity is not constant, but variable with the depth, we have found that the distribution of the velocities with the depth show the same general features as before, the only effect being that the changes are more rapid when the eddy viscossity is small, and slower when the eddy viscossity is great. This is illustrated by the fact that, when the variations of ν are small, we may transform the differential-equation to a form which only differs from the equation of constant ν by terms of second order. This is done by introducing the variable

$$x = \sqrt{\nu} \int_0^z \frac{dz}{\sqrt{\nu}}.$$

From this expression it is seen that when ν is small x increases more rapidly and when ν is greater, the increase of x is slower.

When the surface is free, the distribution of the velocities does not differ materially from that which we find in the case of constant eddy viscossity, but when the surface is ice-covered, the variability of ν is of greater importance. It has been shown that the configuration of the motion may be quite different from that which we find in the case of constant eddy viscossity.

The progressive waves, here considered, have an apparent velocity of propagation which is generally greater than $\sqrt{gh}$. This apparent velocity of propagation depends on the ratio $s = \frac{\lambda}{\sigma}$ and also on the quantity θ, which determines the slope of the wavecrests.

The decrease of the amplitude of the wave has two different causes, primarily the dissipation of energy due to the eddy viscossity, and secondly a complex effect of the viscossity and the rotation. The dissipation of energy can not therefore be computed by means of the damping of the wave.

The kinetic energy of the wave consists of a constant part and a variable part. When the viscossity is left out of account, the variable part of the kinetic energy is equal to the potential energy.

The reflection of the tidal wave is considered in one simple case, when the wave is independent of y and the shore is perpendicular to the direction of propagation. It is found that the orientation of the current-ellipses is unchanged, but the phase-difference between current and surface-elevation is increased. The phase-difference is greatest near the shore, at greater distances from the shore the phase-difference is the same as for the incident wave alone.

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